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RESPONSE TO LÉVY NOISE AND FLUCTUATION-DISSIPATION RELATION

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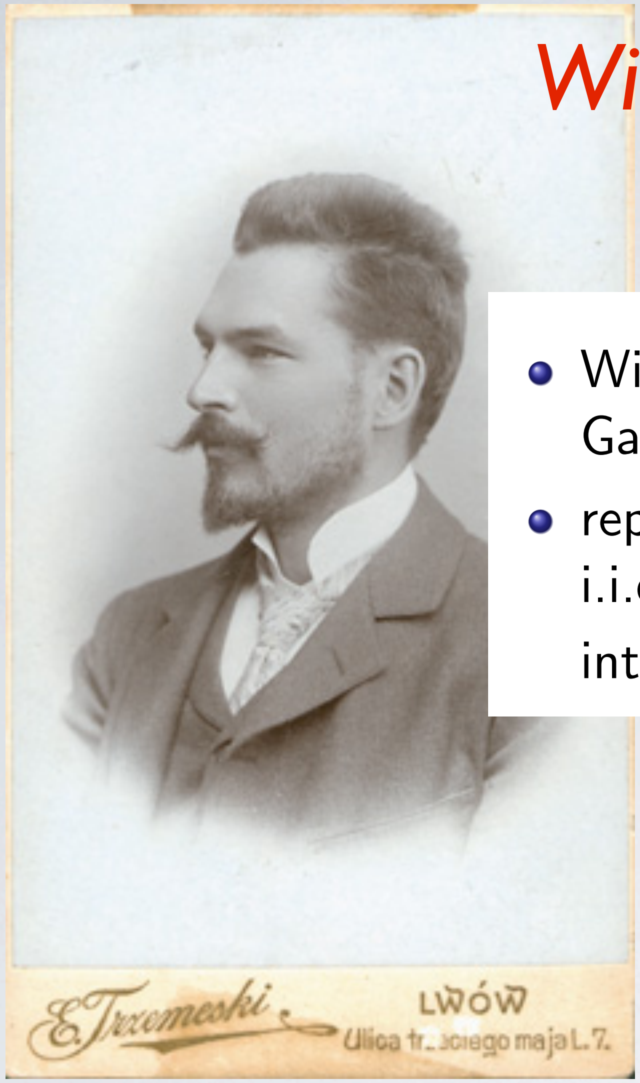
KNOW Krajowy Naukowy Ośrodek Wiodący

Motivation

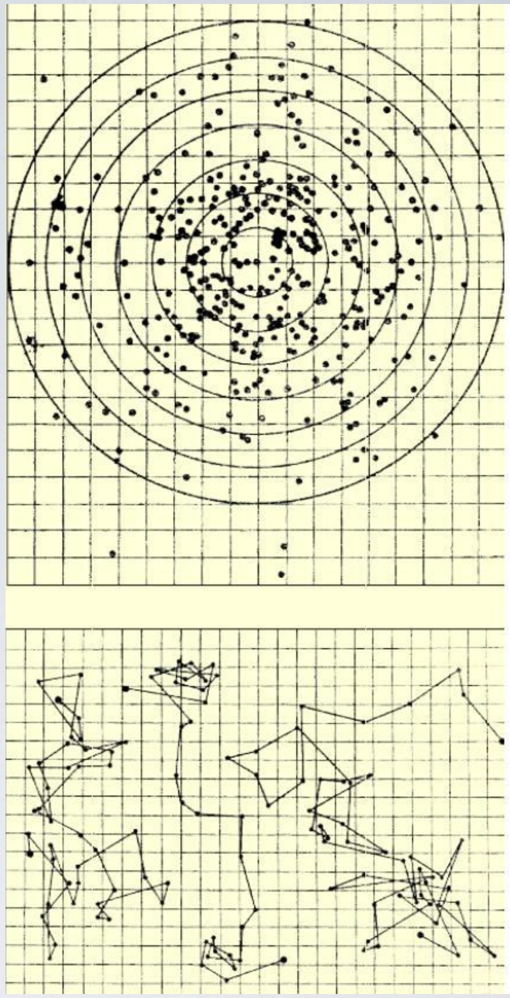
- *Lévy statistics: implies new properties departing strongly (quantitatively and qualitatively) from standard statistical behaviors*
- *Have been addressed in various fields: dynamics in plasma, self- diffusion in micelle systems, exciton and charge transport in random polymers under conformational motion, laser cooling and coherent radiation trapping, analysis of complex systems...*
- *Nonequilibrium noises - non-Gaussian character, in general, space and time correlated*
- *Thermodynamics in the presence of non-Gaussian fluctuations ?*

Wiener process and Brownian motion



Marian Smoluchowski (1872-1917)

- Wiener process $W(t)$: stationary with independent and Gaussian-distributed increments, $\langle W(t)W(s)\rangle = \min(t,s)$
- representation of a Brownian motion: a limit in distribution of i.i.d (Gaussian) jumps taken at infinitesimally short time intervals of random length $W(t) = \lim_{n\to\infty} \sum_{k=1}^{N(nt)} X_k$



$$\frac{\partial p(x,t)}{\partial t} = \sigma^2 \frac{\partial^2 p(x,t)}{\partial x^2}$$

$$p(x,t) = \frac{1}{2\sqrt{\pi\sigma^2 t}} e^{-x^2/(4\sigma^2 t)}$$

$$m\ddot{x} + U'(x) + \eta\dot{x} = \xi(t)$$

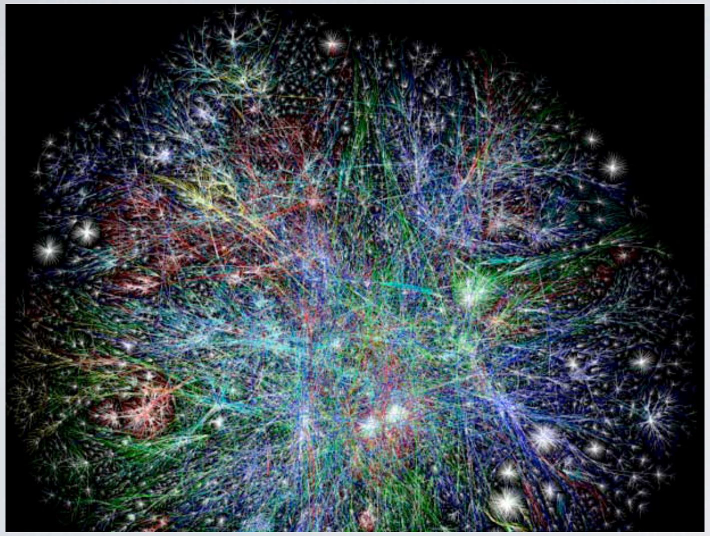
$$\langle \xi(t)\xi(t')\rangle = 2\Gamma\delta(t-t'), \langle\xi(t)\rangle = 0$$

$$\Gamma = \eta k_B T, \langle x^2(t)\rangle \stackrel{t\to\infty}{\to} \frac{2k_B T}{m\eta} t$$

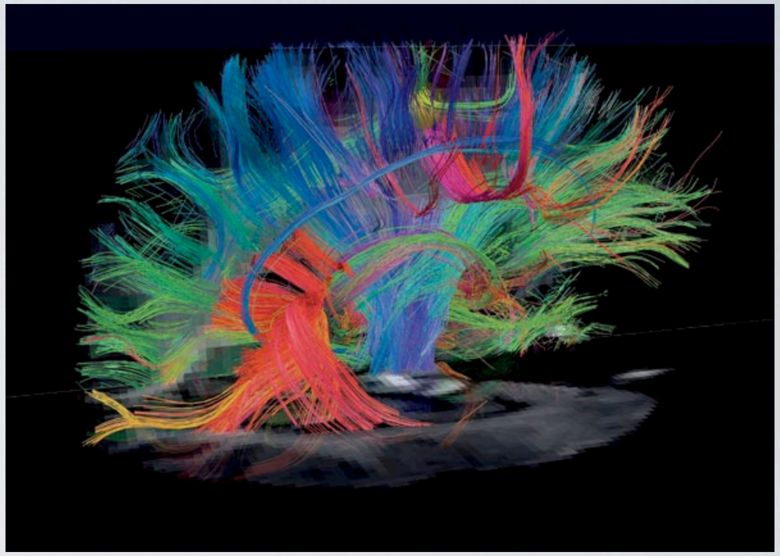
$$\langle x^2 \rangle = 2\sigma^2 t$$

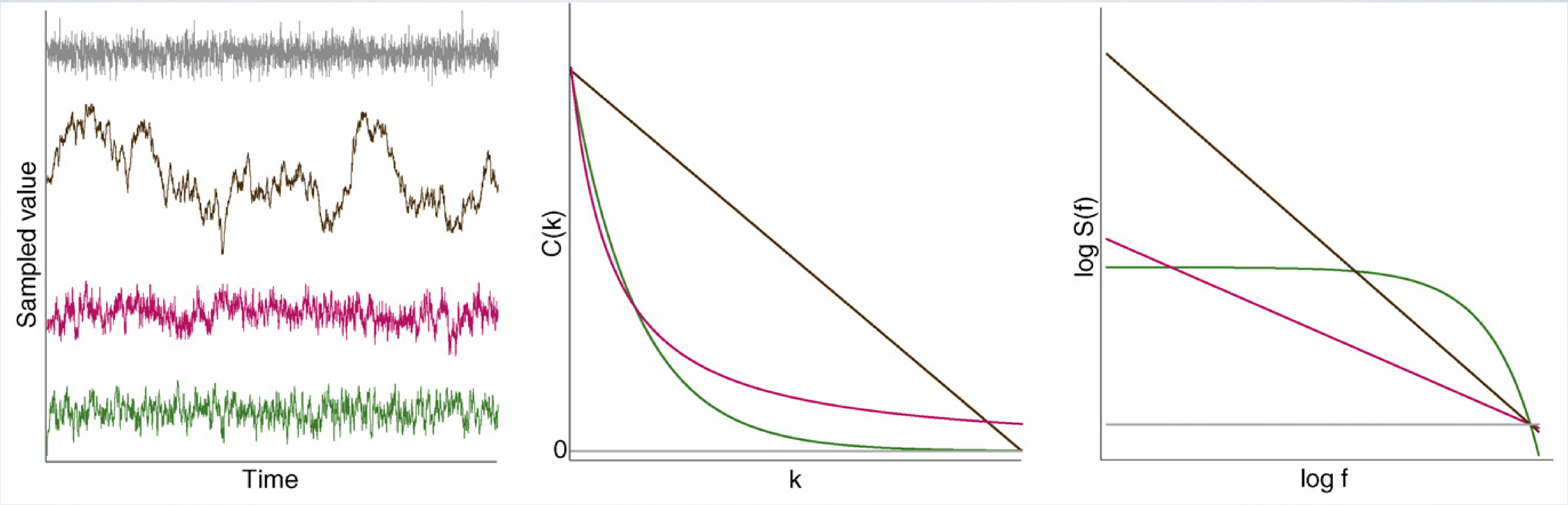
strength of fluctuations related to the magnitude of dissipation

Scaling laws:



complex systems - time/space series analysis





time series *correlation function* *power spectrum*

Kello et al, Trends in Cognitive Sciences, 14, p.223-232 (2010)

power laws and non-Gaussian fluctuations are ubiquitous in Nature!

pink noise

$$S(f) = const \times f^{-\alpha}$$

Non-Gaussian stable white noise

- Generalized Wiener process $W_{\alpha,\beta}(t)$ – non-Gaussian, with stationary and independent increments distributed according to the α-stable law

$$W_{\alpha,\beta}(t) = \int_0^t \zeta(s)ds = \int_0^t dL_{\alpha,\beta}(s) \approx \sum_{i=0}^{N-1} (\Delta s)^{1/\alpha}\zeta_i,$$

ζ_i: i.i.d variables with the stable Lévy probability density function $l_{\alpha,\beta}(\zeta)$, $N\Delta s = t - t_0$, asymptotics $l_{\alpha,\beta}(\zeta) \backsimeq |\zeta|^{-1-\alpha}$.

$$\phi_\zeta(k) = \int_{-\infty}^{+\infty} d\zeta e^{ik\zeta} l_{\alpha,\beta}(\zeta;\sigma,\mu)$$

$$= \exp\left[-\sigma^\alpha |k|^\alpha \left(1 - i\beta \operatorname{sign} k \tan\frac{\pi\alpha}{2}\right) + i\mu k\right]$$

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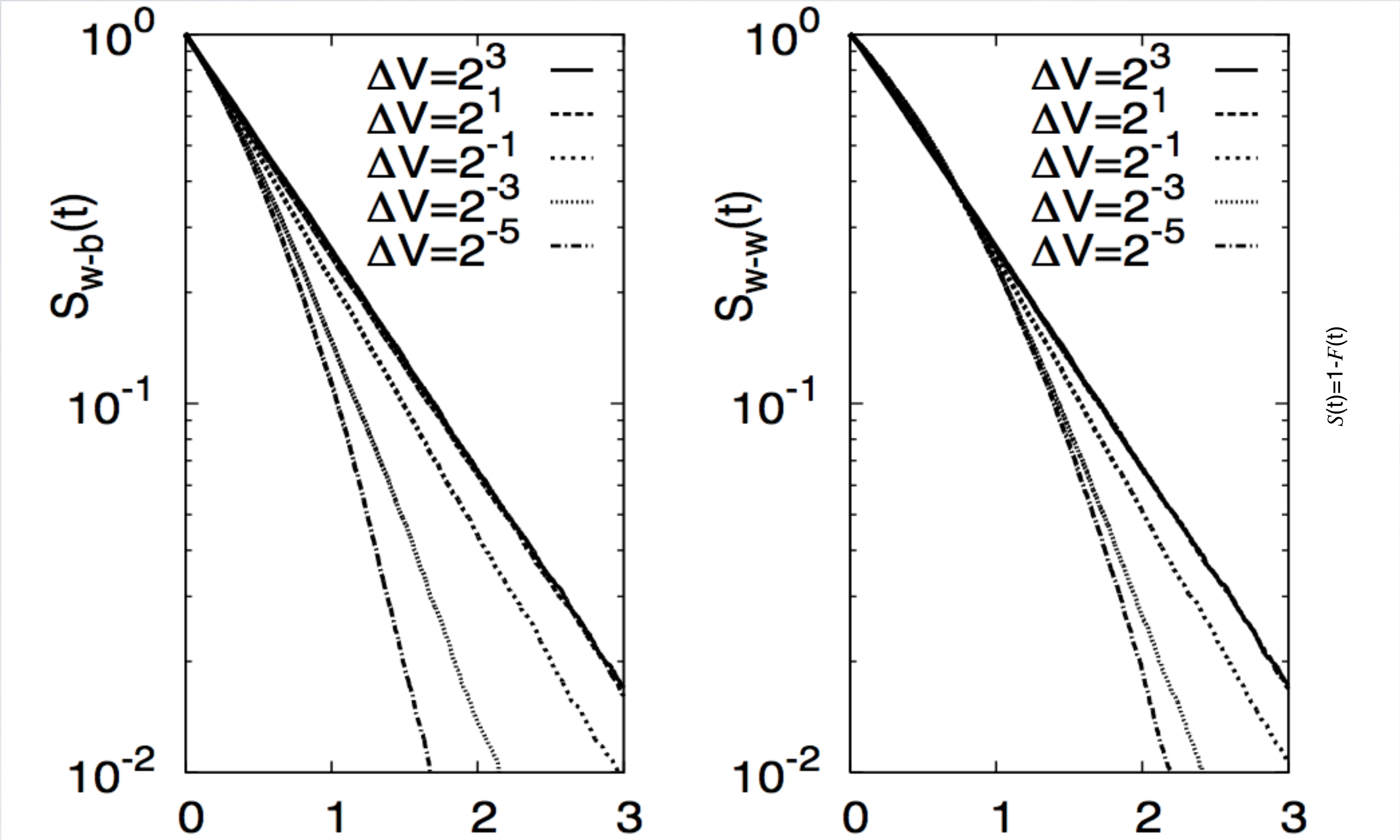
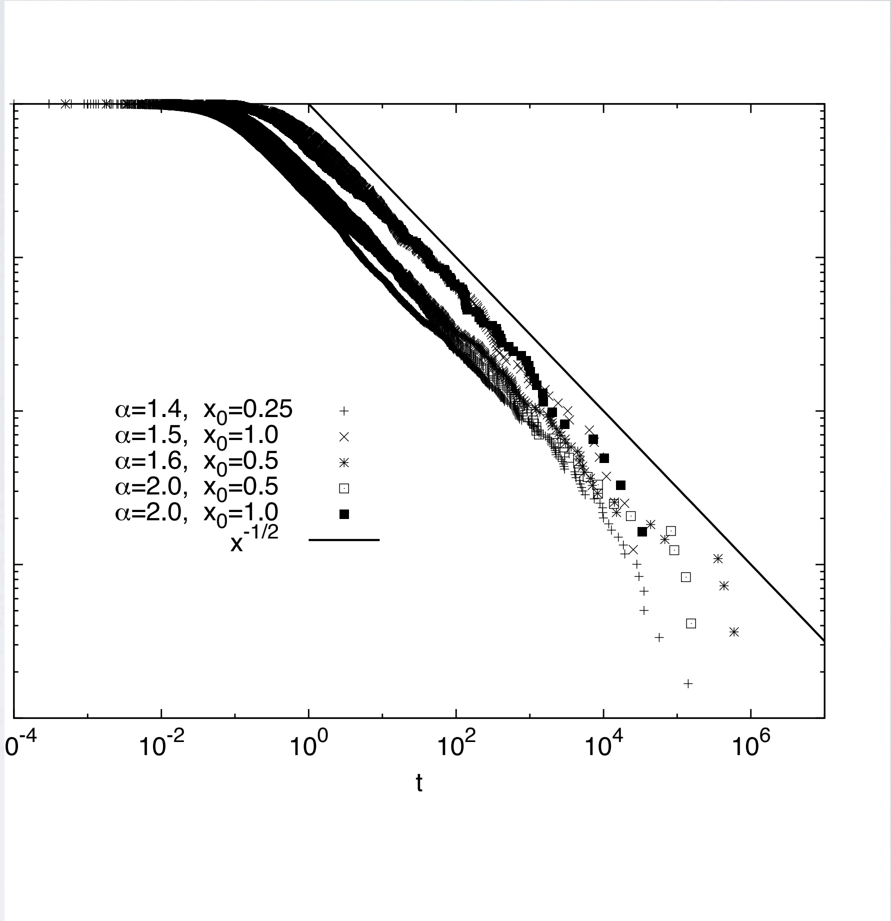
Survival function and probability density of escape times

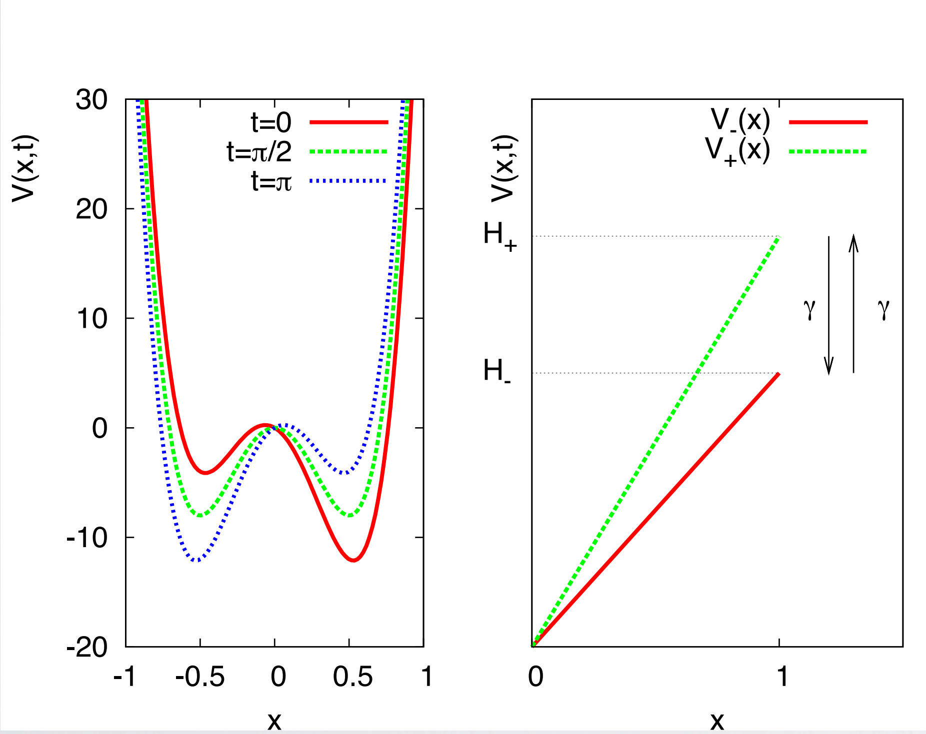




Free diffusion -- escape on a semi-line:
Sparre-Andersen scaling

Following the Markovian character of the stochastic dynamics—at sufficiently high barriers—the time dependence of the survival probabilities within the potential well assume an exponential law

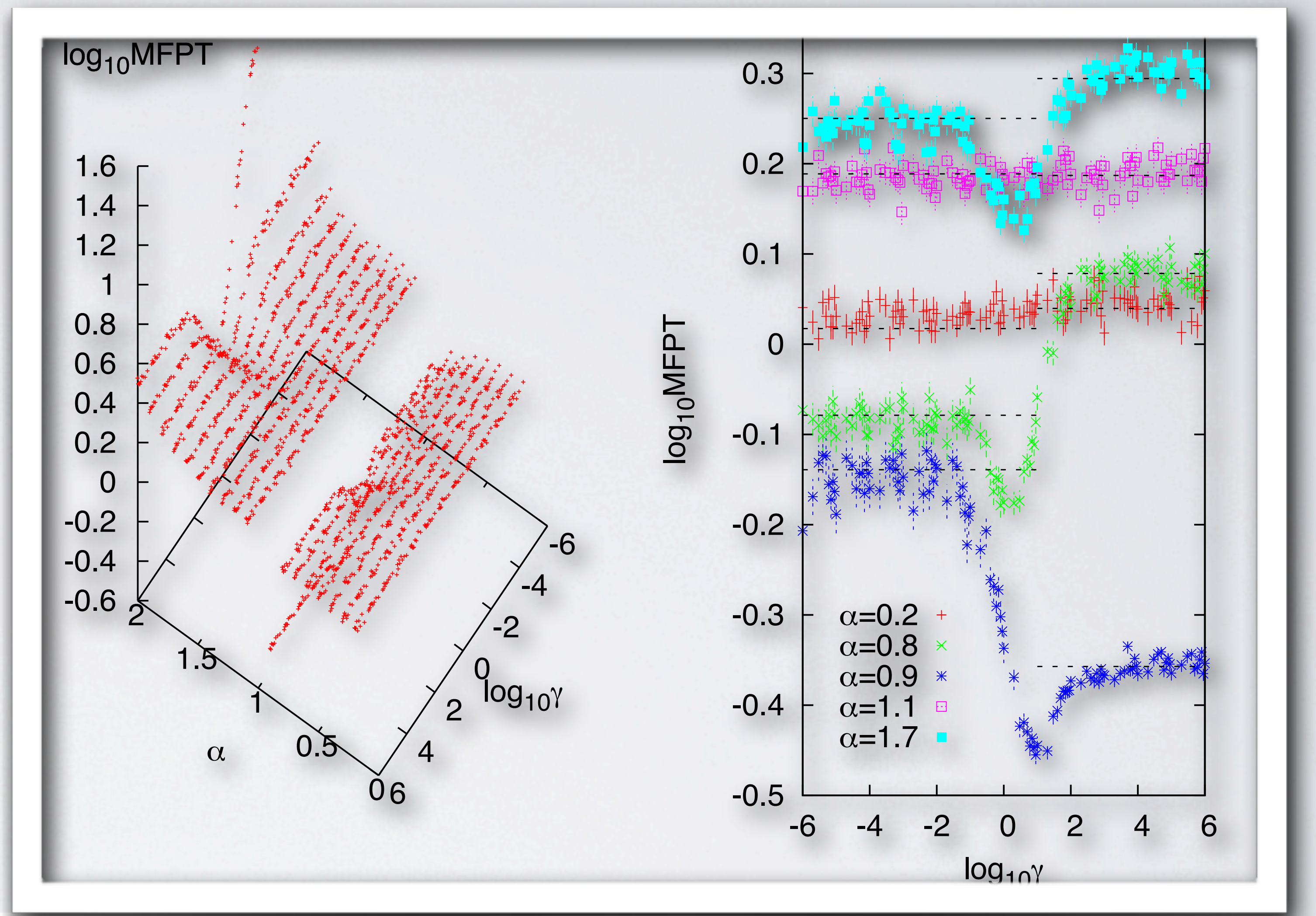
*B.Dybiec, E.G-N, P. Hänggi, PRE, **73**, 046104 (2006)*
*B.Dybiec, E.G-N, P. Hänggi, PRE, **75**, 021109 (2007)*



Fine-tuning to Lévy-white noises: resonant activation and stochastic resonance

- Langevin equation

$$\dot{x}(t) = -V'(x,t) + \zeta(t) \quad \Rightarrow P(x,t)$$



- Fokker-Planck equation

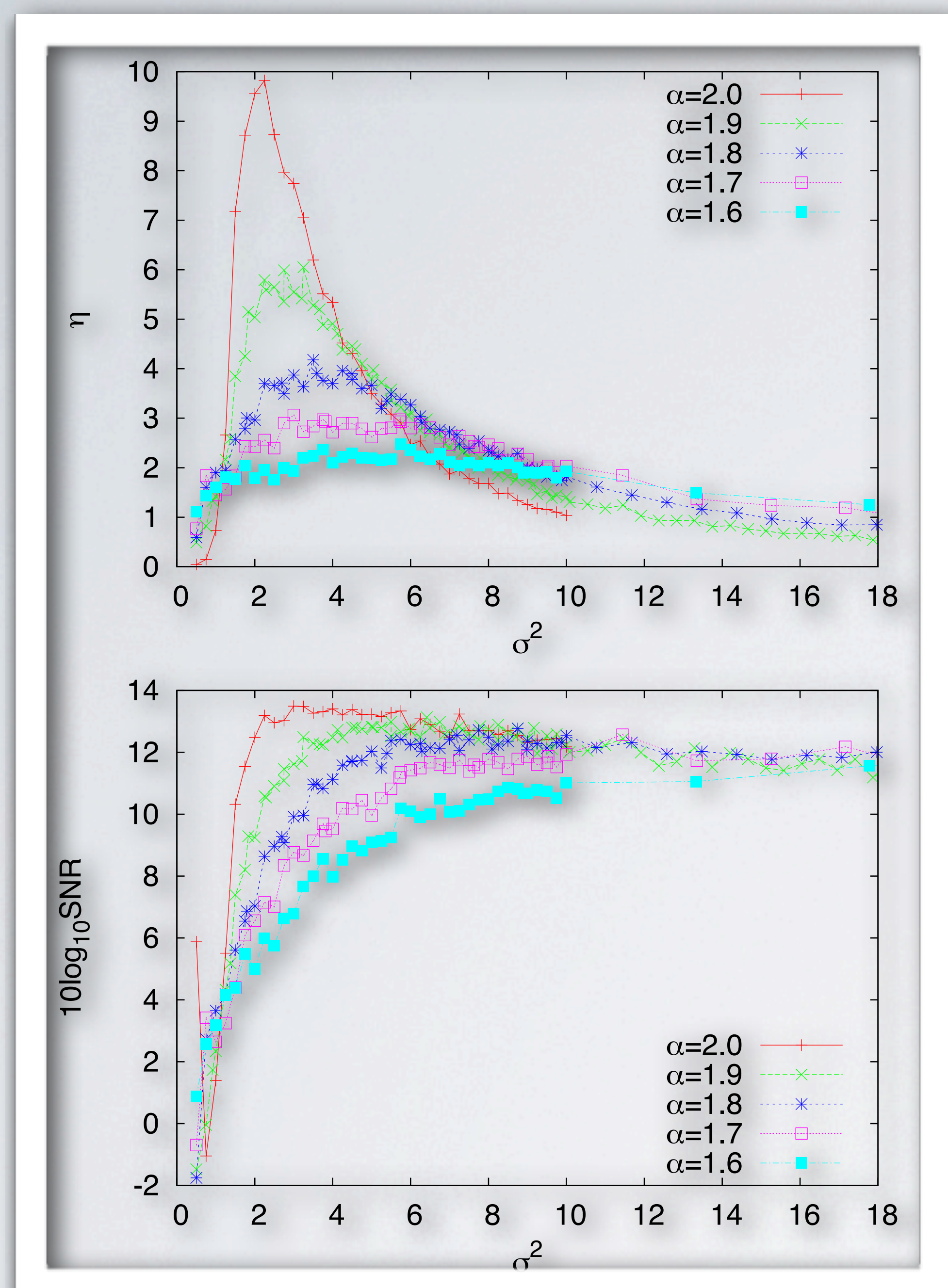
$$\frac{\partial P(x,t)}{\partial t} = \left[\frac{\partial}{\partial x} V'(x,t) + D\frac{\partial^2}{\partial x^2}\right] P(x,t)$$

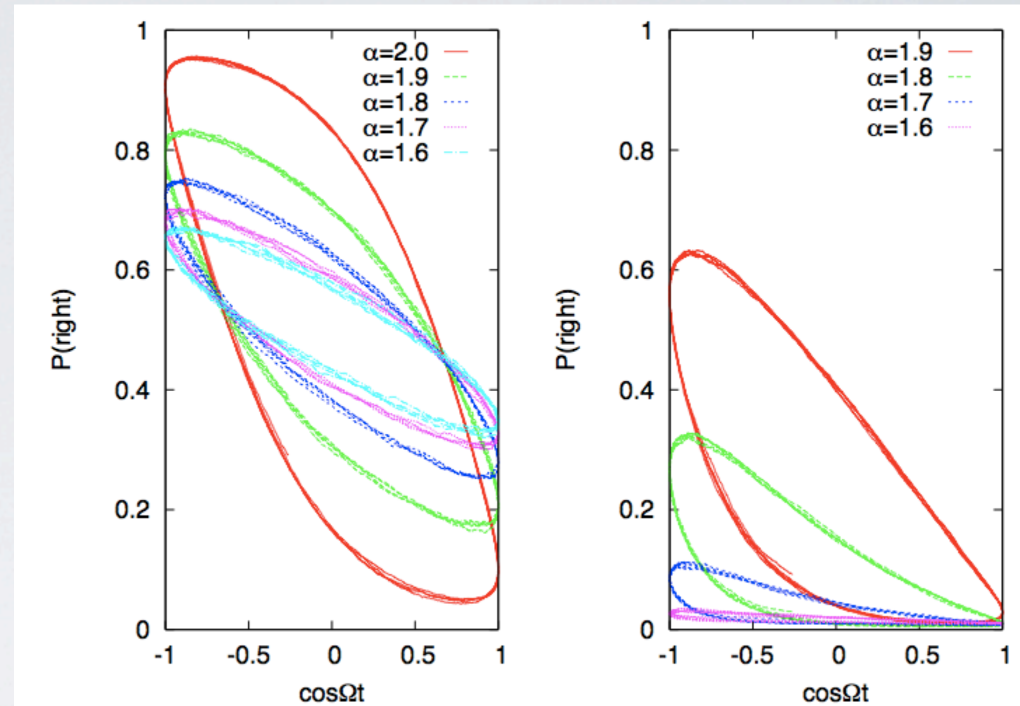
- fractional Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x} V'(x) P(x,t) + \sigma^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}, \text{ where}$$

$$\frac{\partial^\alpha}{\partial |x|^\alpha} f(x) = -\int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{-ikx} |k|^\alpha \hat{f}(k) \text{ and } D = \sigma^\alpha.$$

Stochastic resonance and dynamical hysteresis





B. Dybiec, E.G-N J. Stat. Mech. P05004 (2009)
B. Dybiec, E.G-N, New J. Phys. ***9,*** *452 (2007)*
B. Dybiec, E.G-N, I.M. Sokolov, PRE ***78*** *011117 (2008)*

$\varsigma_i \sim S_\alpha(\sigma, \beta, \mu = 0)$ and $(N-1)\Delta s = t$.

$$x(t) = -\int_0^t V'(x(s), s)ds + \int_0^t dL_{\alpha,\beta}(s)$$
$$\approx -\int_0^t V'(x(s), s)ds + \sum_{i=0}^{N-1} \Delta s^{1\alpha} \varsigma_i$$

Action of Lévy type noises

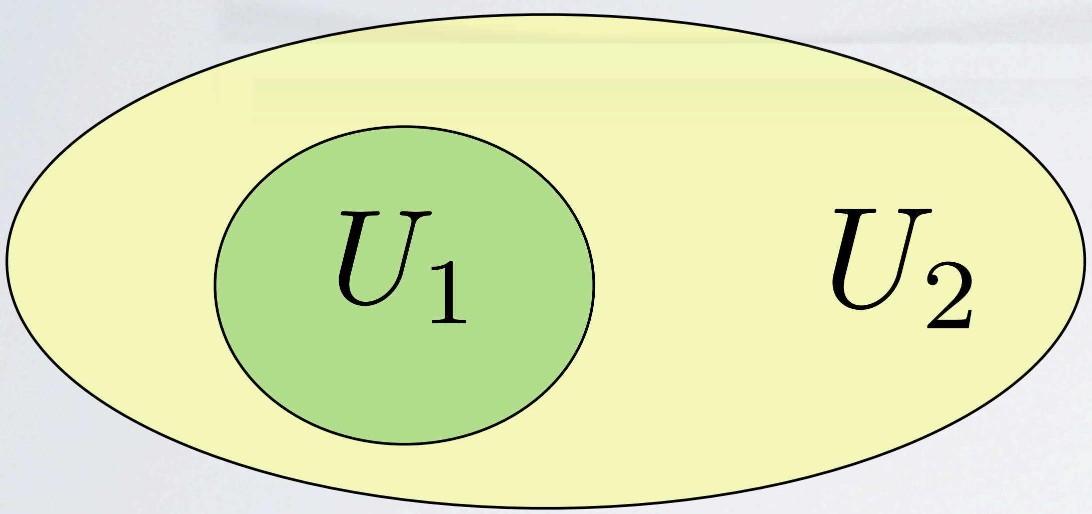
- *Systems embedded in noisy environments may enhance sensitivity*
- *Counterintuively, despite their pathological character (diverging moments) Lévy fluctuations may induce better signal transmission*

- *Lévy white noises acting in nonlinear dynamic systems exhibit positive, ordering effects: stochastic resonance, resonant activation, synchronization and directionality of transport (ratcheting effect)*

Equilibrium conditions and linear response...

$$S_\nu(E_{\nu 1}, E_{\nu 2}, \ldots)$$

$$S_{total} = \sum_\nu S_\nu(E_{\nu 1}, E_{\nu 2} \ldots)$$



$U_1 \oplus U_2$ *isolated*

$$E_j = E_{1j} + E_{2j} = const$$

conditions

$$\delta S_1(E_{1j}) + \delta S_2(E_{2j}) = 0 \qquad \delta E_{1j} + \delta E_{2j} = 0$$

$$\Rightarrow \left(\frac{\partial S_1}{\partial E_{1j}} - \frac{\partial S_2}{\partial E_{2j}} \right) \delta E_{1j} = I_{1j} - I_{2j} \equiv 0 \qquad \forall \delta E_{1j}$$

Nonequilibrium and linear response...

Lars Onsager (1903-1976)

$$X_j \equiv \left(\frac{\partial S_1}{\partial E_{1j}} - \frac{\partial S_2}{\partial E_{2j}} \right) \delta E_{1j} = I_{1j} - I_{2j} \neq 0$$

„thermodynamic forces"

thermodynamic forces generate fluxes

$$\frac{dE_{1,j}}{dt} = \Phi_j$$

In consequence, entropy production given by a product of force and conjugated flux

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial E_{1,j}} \frac{dE_{1j}}{dt} = \sum_j X_j \Phi_j$$

Onsager (~1932) theory for weak X forces forsees...

$$\Phi_j = \sum_k L_{kj} X_k \Rightarrow \frac{dS}{dt} \approx \sum_k L_{kj} X_k X_j$$

with the IInd law

$$\frac{dS}{dt} \geq 0$$

D. Reguera, J.M. Rubi, J.M.G. Vilar J.Phys. Chem B **109** 21502 (2005)

Nonequilibrium conditions and linear response ? fluctuation-dissipation theorem ?

$$f(t) = f_0 \Theta(-t)$$

$$\langle x(t) \rangle = \int dx' \int dx \;\; x' p(x', t|x, 0) \tilde{p}(x, 0)$$

$$\tilde{p}(x, 0) = \frac{e^{-\beta H(x)}}{\int dx' e^{-\beta H(x')}} = \frac{e^{-\beta [H_0(x) + x f_0]}}{\int dx' \; e^{-\beta H(x')}}$$

weak perturbation

$$e^{-\beta x f_0} \approx 1 - \beta x f_0$$

$$\tilde{p}(x, 0) \approx p_0(x)(1 - \beta f_0 x)$$

$$\boxed{\langle x(t) \rangle} = \int dx' \int dx \;\; x' p(x', t|x, 0) p_0(x)(1 - \beta f_0 x) =$$

$$\boxed{\langle x \rangle_0 - \beta f_0 \langle x(t) x(0) \rangle_0}$$

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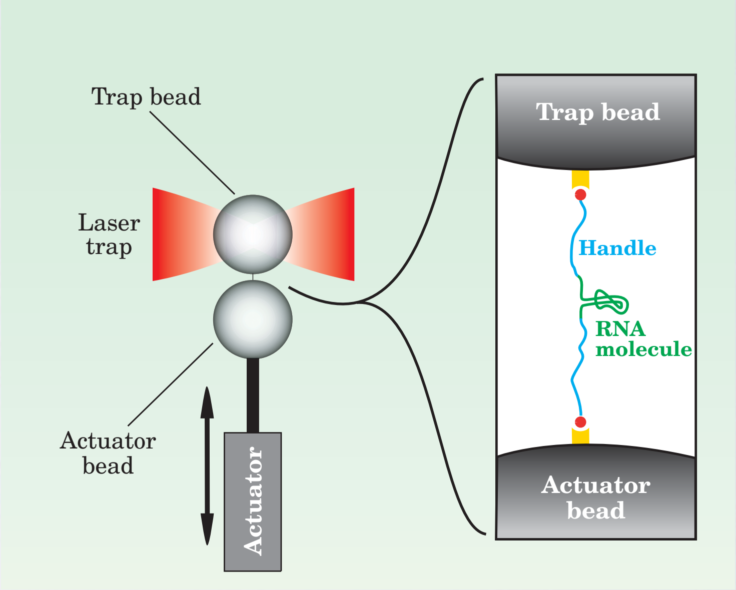
Linear response, fluctuation-dissipation theorem?

On the other hand...

$$\langle x(t)\rangle = \langle x\rangle_0 + \int_{-\infty}^{t} f(\tau)\chi(t-\tau)d\tau$$

$$f_0 \int_0^{\infty} d\tau\Theta(\tau - t)\chi(\tau) = \beta f_0 \langle x(t)x(0)\rangle_0$$

$$-\chi(t) = \beta\frac{d}{dt}\langle x(t)x(0)\rangle_0$$



F. Ritort et al, Phys. Today, **7** *43 (2005)*

FDT relates susceptibility to correlations measured in the reference unperturbed state

reference stationary state

$$\langle A(t)\rangle - \langle A\rangle_0 \simeq \int_0^t \chi_{A,\gamma}(t-t')\delta\lambda_\gamma(t')dt', \quad (1)$$

$$\chi_{A,\gamma}(t-t') = \frac{d}{dt}\langle A(t)X_\gamma(t')\rangle_0, \quad (2)$$

$$X_\gamma(x) = -\left.\frac{\partial \ln \rho_{ss}(x;\vec{\lambda})}{\partial \lambda_\gamma}\right|_{\vec{\lambda}=\vec{\lambda}_0} = \frac{\partial \phi}{\partial \lambda_\gamma}. \quad (3)$$

*J. Prost, JF Joanny, J.M.R. Parrondo, PRL **103** 090601 (2009)*
*U. Seifert, T. Speck, Europhys. Lett. **89** 1007 (2010)*
*B. Dybiec, J.M.R. Parrondo, E.G-N Europhys. Lett **98** 50006 (2012)*

$$\phi \equiv -\ln \rho_{ss}$$

$$\rho_{ss}(x;\vec{\lambda}) = \frac{\exp[-\beta\mathcal{H}(x;\vec{\lambda})]}{Z(\beta,\vec{\lambda})}$$

$$X_\gamma(x) = \frac{1}{kT}\left[\frac{\partial\mathcal{H}(x;\vec{\lambda}_0)}{\partial\lambda_\gamma} - \left\langle \frac{\partial\mathcal{H}(x;\vec{\lambda}_0)}{\partial\lambda_\gamma}\right\rangle_0\right]$$

FDT: measurable macroscopic physical quantities related to correlations functions of spontaneous fluctuations

$$X_\gamma(x) = \frac{1}{kT}\left[\frac{\partial \mathcal{H}(x;\vec{\lambda}_0)}{\partial \lambda_\gamma} - \left\langle \frac{\partial \mathcal{H}(x;\vec{\lambda}_0)}{\partial \lambda_\gamma} \right\rangle_0\right]$$

$$X_\gamma(x) = \frac{1}{kT}\frac{\partial\left[\mathcal{H}(x;\vec{\lambda}) - F(\beta,\vec{\lambda})\right]}{\partial \lambda_\gamma}\Bigg|_{\vec{\lambda}=\vec{\lambda}_0}$$

$$\rho_{ss} = \exp[-\beta\mathcal{H}]Z^{-1} \qquad T(t) \to T + \delta T$$

$$\delta\mathcal{H} = \alpha(t)\delta T$$

isochoric specific heat estimated by analysing fluctuations in the steady state

$$\alpha(t) = \frac{1}{kT^2}\langle \delta\mathcal{H}(0)\delta\mathcal{H}(0)\rangle_0$$

Conjugate variable...

$$\begin{cases} \dot{x}(t) = -ax + f(t) + \zeta(t) \\ x(0) = x_0 \end{cases}$$

$$\langle X(t) \rangle = \int_{-\infty}^{\infty} X(x) p(x,t) dx$$

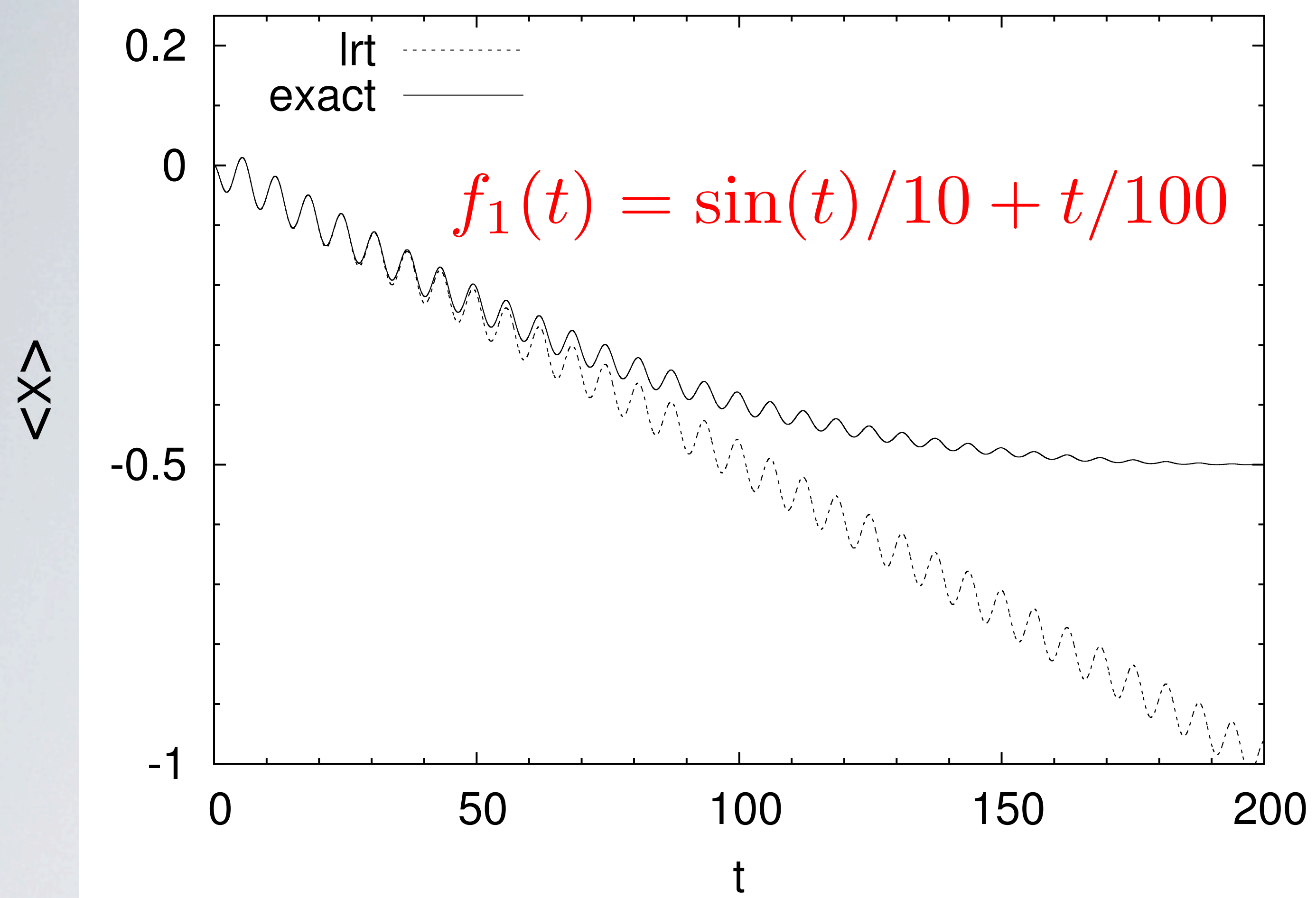
$$\hat{p}(k,t) = \exp\left[ik\mu(t) - \sigma^\alpha(t)|k|^\alpha \left(1 - i\beta\,\mathrm{sign}(k)\tan\frac{\pi\alpha}{2}\right)\right]$$

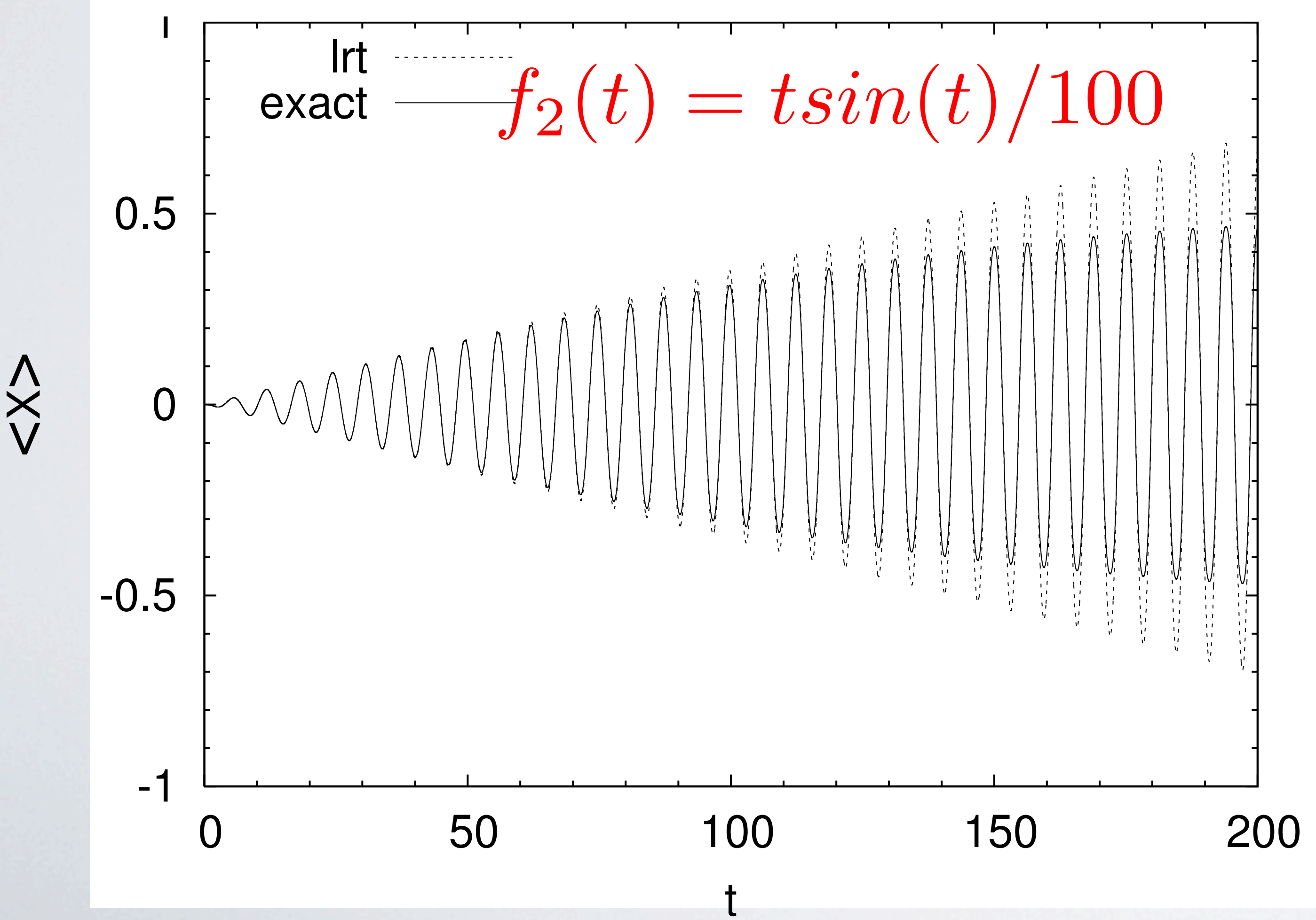
$$p_{1,0}(x,t|x_0,0) = \frac{\sigma(t)}{\pi}\frac{1}{[x-\mu(t)]^2 + \sigma^2(t)}$$

$$X_{\mathrm{C}} = -\frac{2x}{a\left[x^2 + (\sigma_0/a)^2\right]}$$

$$\chi(t) = \frac{d}{dt}\langle X(t)X(0)\rangle_0 = -\frac{a}{2\sigma_0^2}e^{-at}$$

$$\langle X(t) \rangle_{\mathrm{LR}} = \int_0^t \chi(t-s) f(s) ds$$





***Response** of conjugate variable to external drivings:*
solid lines -- exact result
dotted lines -- result constructed by use of the linear response theory

Exact <X> for a constant force:

$$\langle X \rangle = -\frac{\sigma_0}{a\pi}\int_{-\infty}^{\infty} \frac{dx}{[x-f/a]^2+(\sigma_0/a)^2}\,\frac{2x}{a\,[x^2+(\sigma_0/a)^2]} = -\frac{2f}{f^2+4\sigma_0^2}$$

***Conjugate** variables reflect change in the PDF under the perturbation*

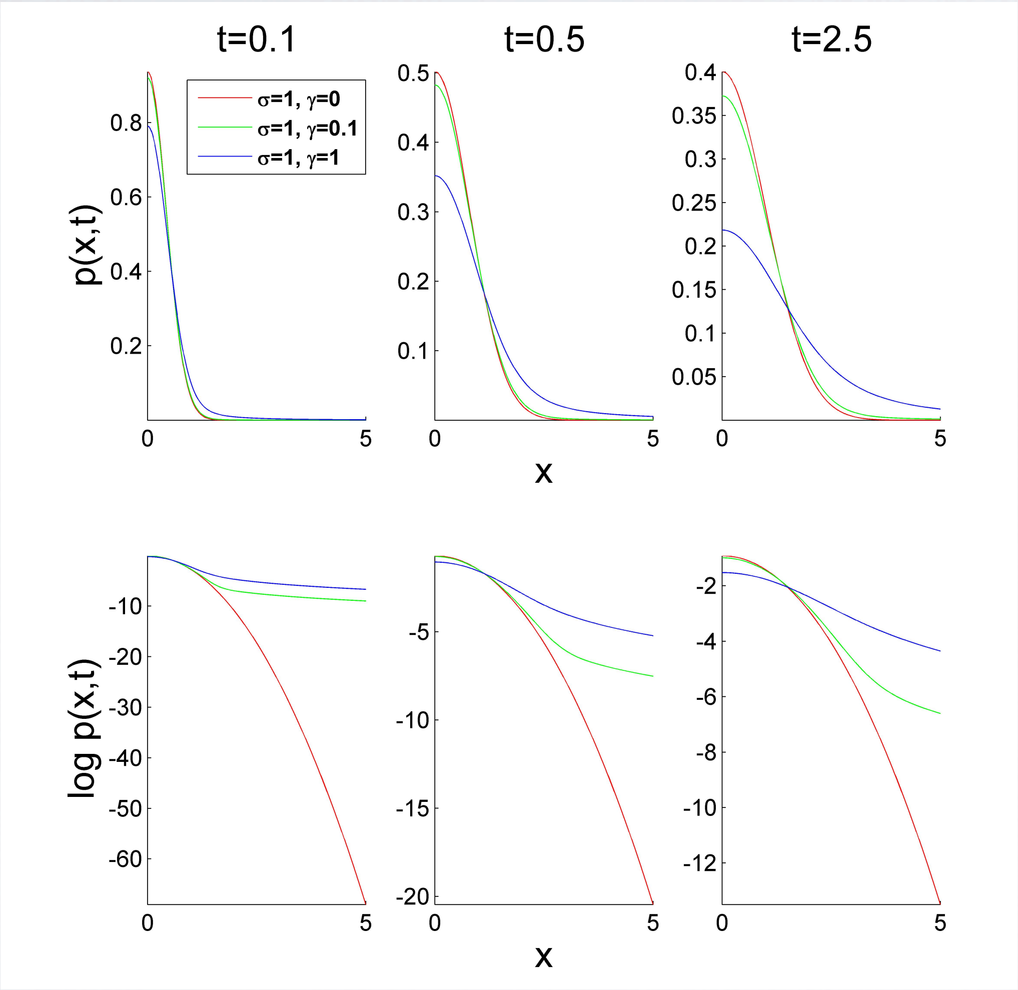
*Despite the system is plagued by **divergent moments,** the generalized FDT properly captures dynamical response*

Drawbacks: interpretation of <X>

Two independent noises: Cauchy and Gauss

$$\dot{x}(t) = \mu_0 - ax + f(t) + \xi_c(t) + \xi_g(t)$$

$$\hat{p}(k,t) = e^{ik\mu(t) - \sigma^2(t)|k|^2 - \gamma(t)|k|}$$



for f(t)=0

$$p_s(x) = \frac{1}{2\sqrt{\pi}\sigma_\infty} \operatorname{Re} w\left(\frac{-x + i\gamma_\infty}{2\sigma_\infty}\right)$$

$$w(x) := e^{-x^2} \operatorname{erfc}(-ix)$$

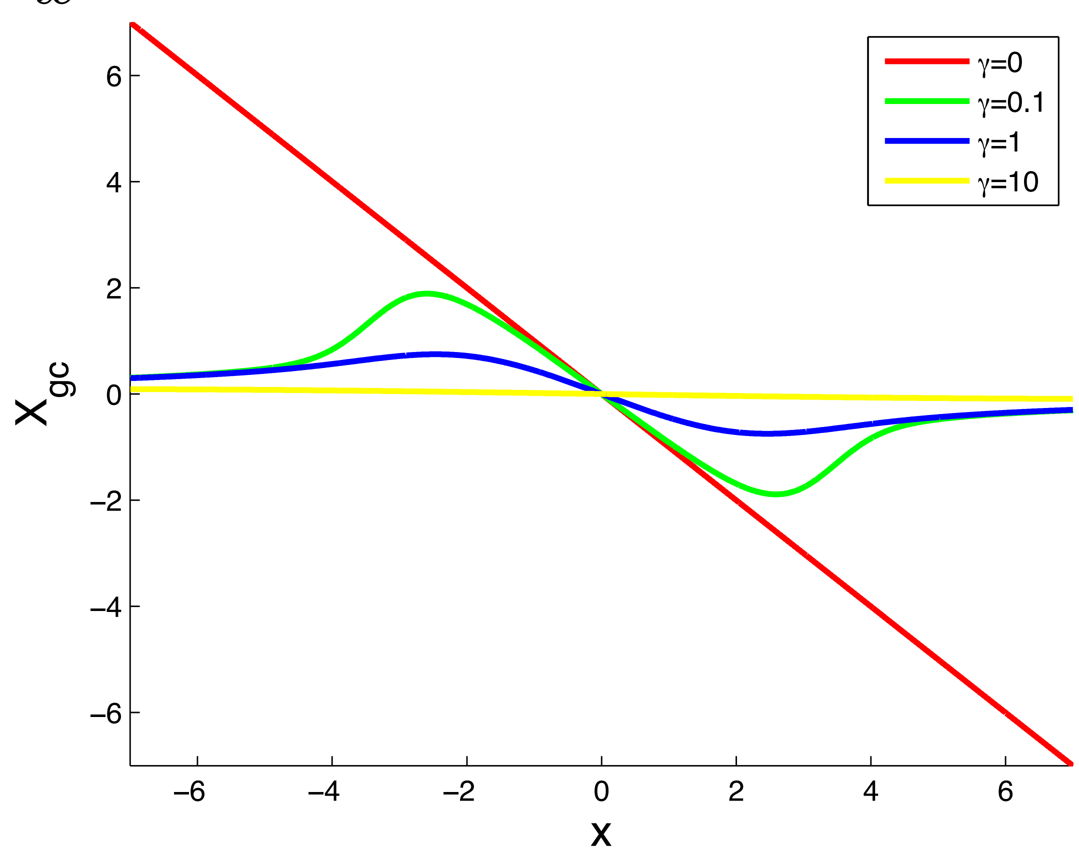
Two independent noises: Gaussian and Cauchy

$$X_{gc} = -\frac{x}{2\sigma_\infty^2 a} - \frac{\gamma_\infty}{2\sigma_\infty^2 a} \frac{\operatorname{Im} w(\frac{-x+i\gamma_\infty}{2\sigma_\infty})}{\operatorname{Re} w(\frac{-x+i\gamma_\infty}{2\sigma_\infty})}$$

$$\lim_{\gamma_0 \to 0} X_{gc} = -\frac{x}{2a\sigma_\infty^2} = -\frac{x}{\sigma_0^2}$$

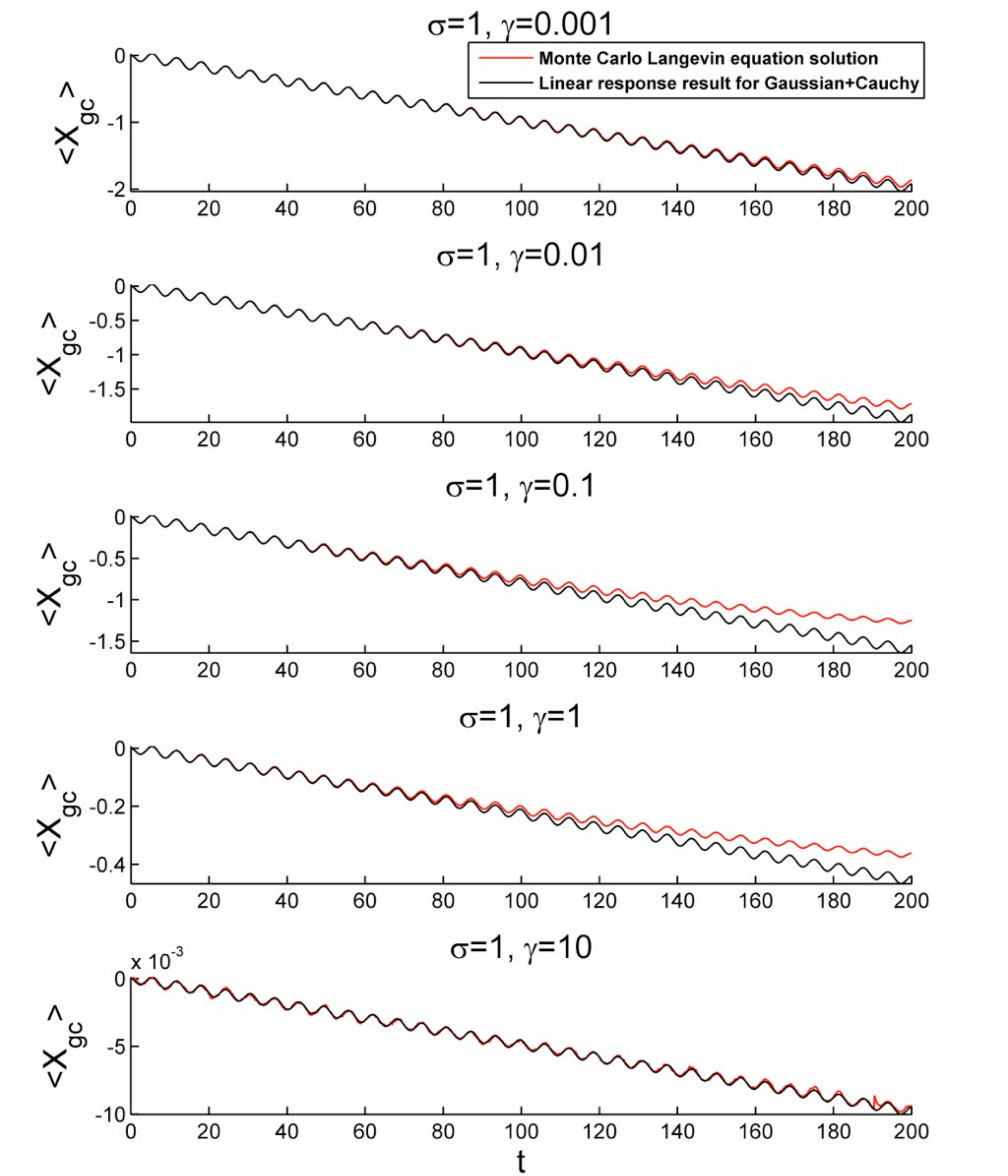
$$\lim_{\sigma_0 \to 0} X_{gc} = -\frac{2ax}{\gamma_0^2 + a^2x^2}$$

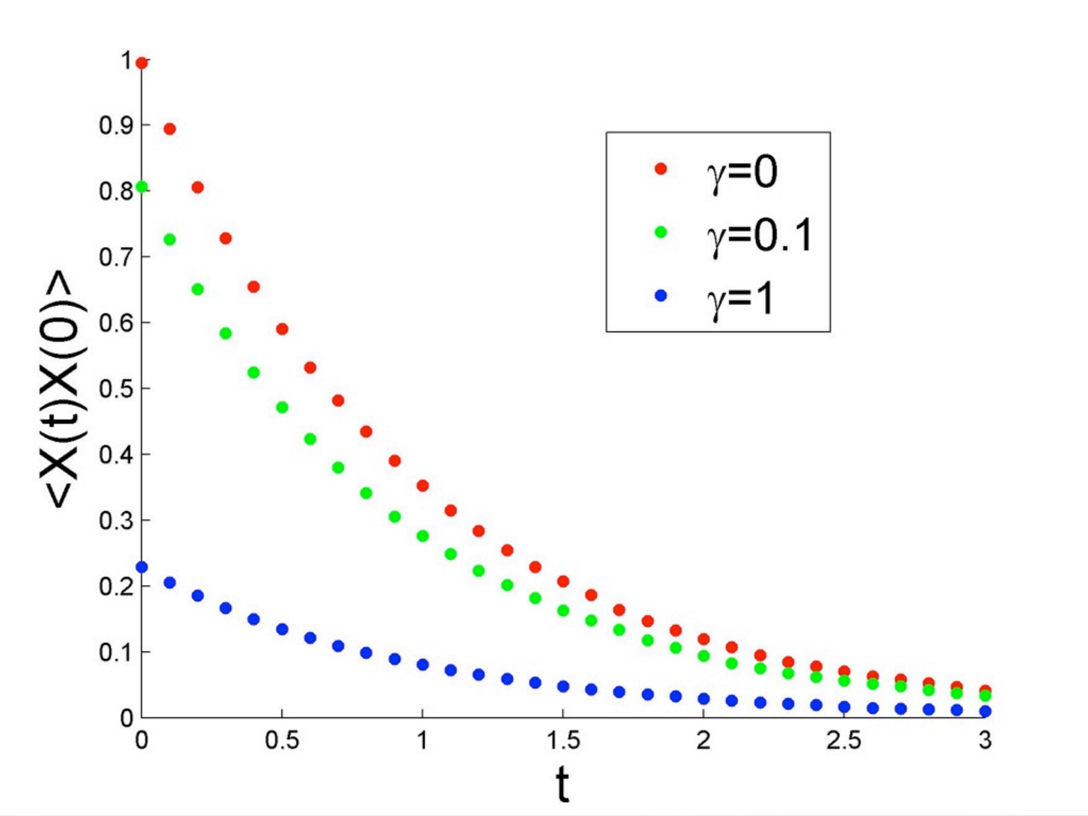
Conjugate variable...



Response and correlation decay

$$f(t) = \sin(t)/10 + t/10$$





- *Nonequilibrium steady states are fascinating systems to study...*
- *The generalized FDT can be applied to (linear) systems driven by Lévy noises*
- *The conjugate variables represent change in PDF under perturbation (in equilibrium related to energy absorbed from perturbations)*
- *Interpretation of conjugate variables is not straightforward...*

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B. Dybiec, W.Ebeling, J. Parrondo, Ł. Kuśmierz

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Exploring Physics of Small Devices
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Breakdown of common thermodynamics?

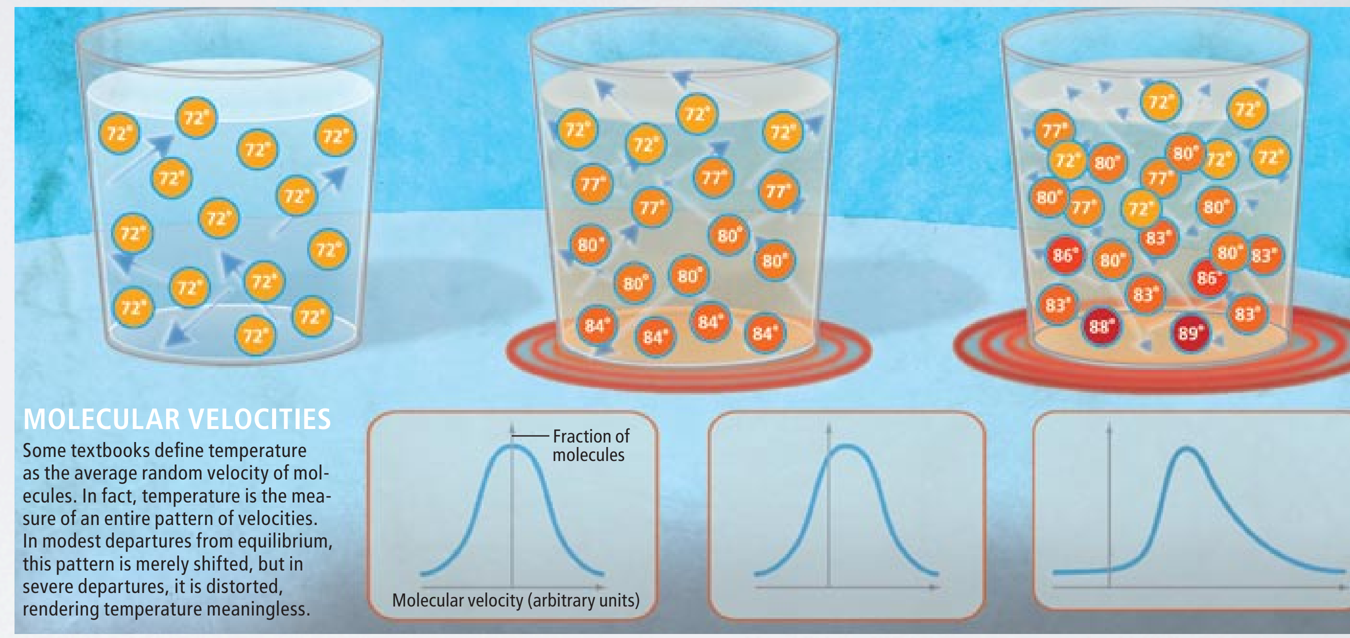
$$dU = \delta Q + \delta W$$

$$\frac{\delta Q}{T} = dS \qquad \Delta S \geq 0$$

In far-from equilibrium situations a common definition of temperature does not make sense
Traditional thermodynamics does not describe transitions between metastable states
Theory is not suitable for description of ordering phenomena in nonequilibrium states

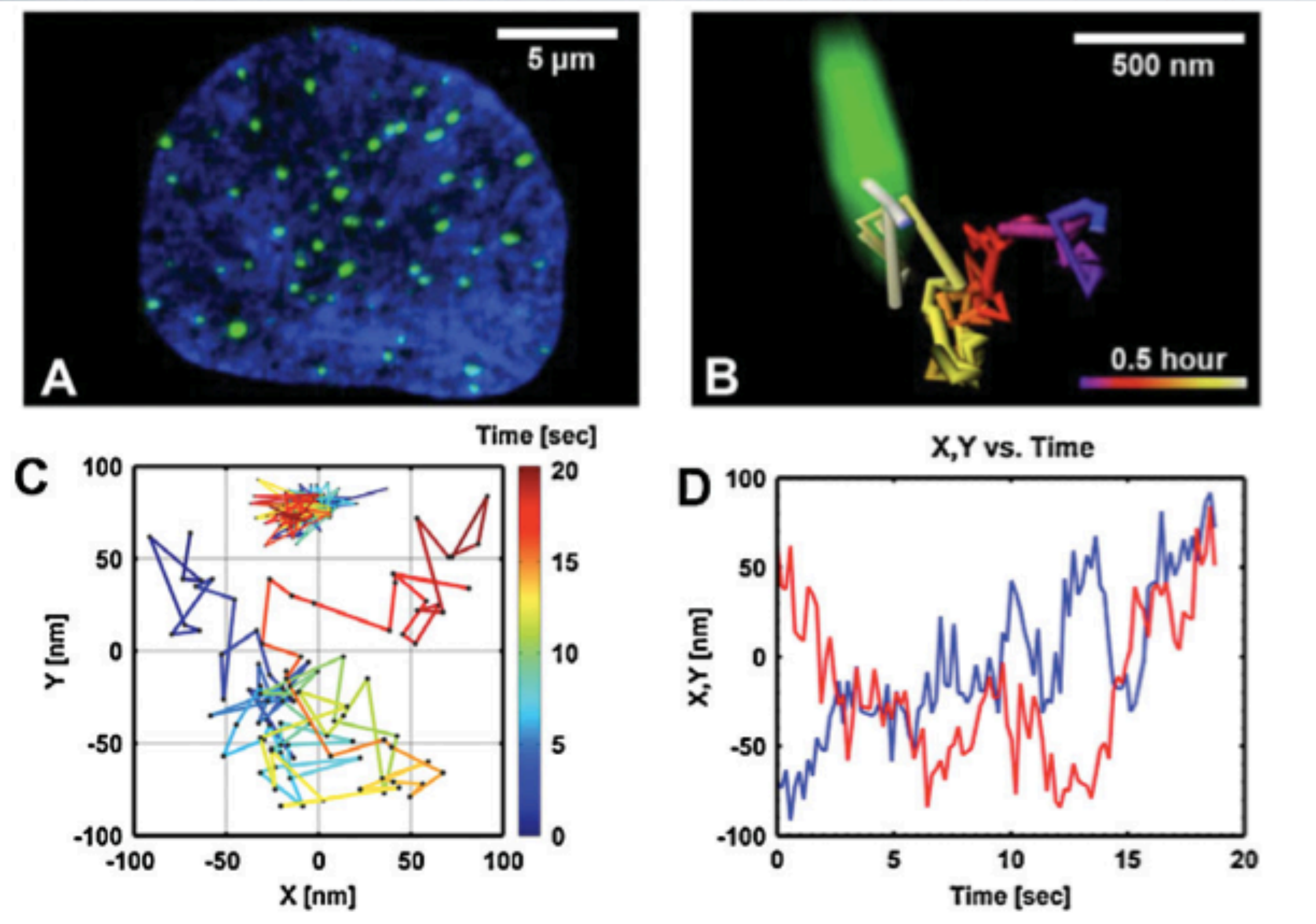
DISSIPATION RELATED TO ORDER!

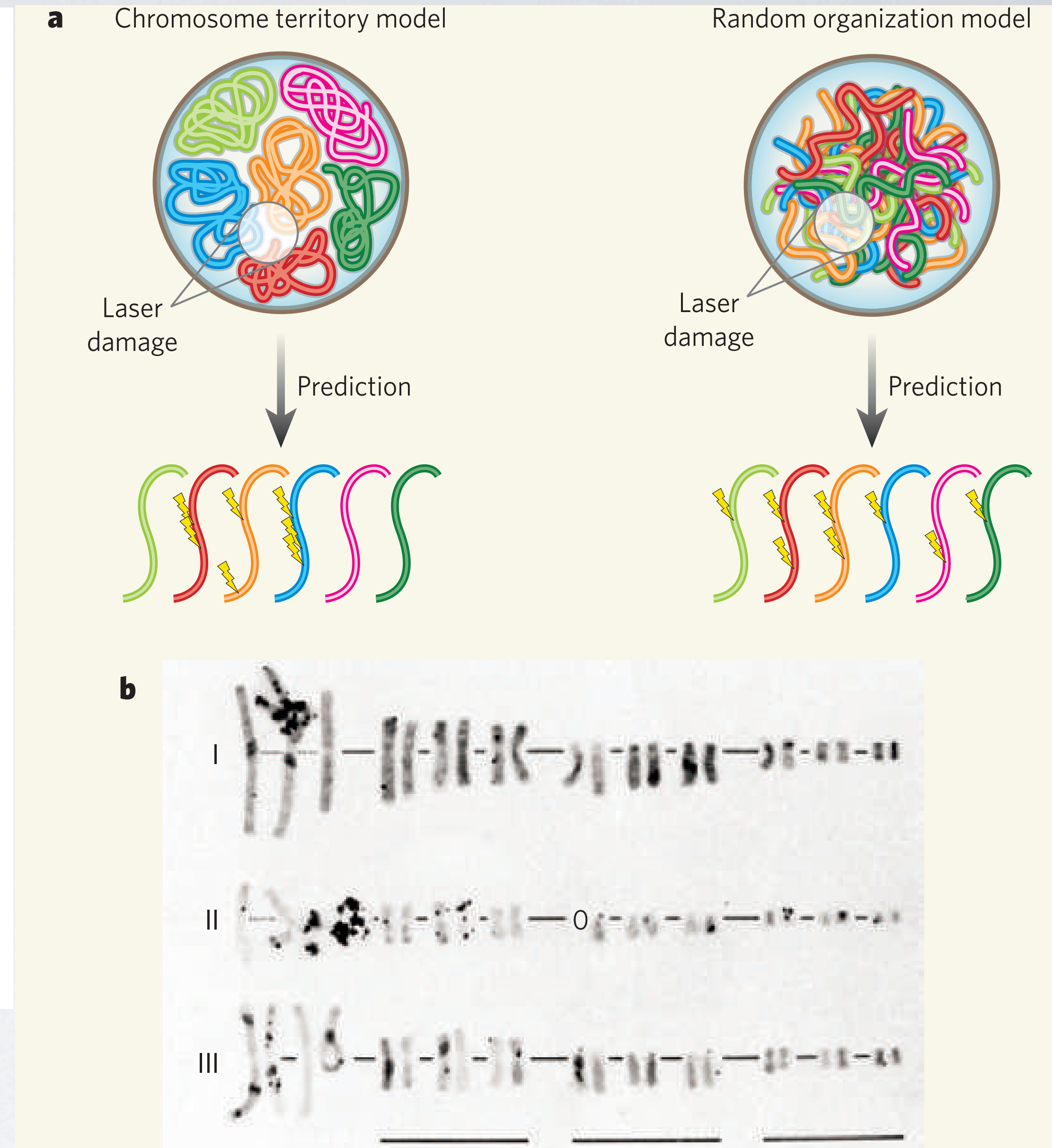
•Axiomatic formulation: „traditional" thermodynamics is an elegant mathematical theory
•Key words: a state, a state-function, equation of state (characteristic of state quantities)
•Infinitesimal, adiabatic changes of state are time-reversible



J.M. Rubi, Scientific American, 2008

Importance of anomalous transport...





*T. Misteli, Nature **445** 379 (2007)*

*I.Bronstein, Y.Israel et al. PRL, **103,** 018102 (2009)*
*R. Klages, G. Radons, I.M. Sokolov **Anomalous Transport** (Wiley VCH, Weinheim, 2008)*
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