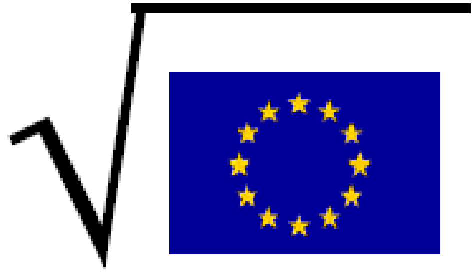
Why square root?
On voting systems in the European Council



Wojciech Słomczyński
Karol Życzkowski
Jagiellonian University (Kraków)

Voting systems for the Council of the European Union

- a two-tier decision-making system:
  - the **Member States** at the lower level
  - the **European Union** at the upper level

The Council of the EU votes by a **qualified majority voting**:
a decision of the Council is taken, if it is approved by a qualified majority

Indirect voting in the Council

- A representative of a member state with a population **N** goes to Brussels and says **yes** according to the will of the majority of his co-patriots...
- How many of them are satisfied, **N** or **N/2** ? (since the representative followed their will).
- ***We do not know!*** These numbers will be different in each cases. Mathematics is needed to compute the average and to prove that the difference **satisfied** - **dissatisfied** scales as ... **Sqrt (N)**

How to analyse voting systems?

- *27* Members States:
  more than ***134 mln*** possible coalitions

- **voting power** (capacity to affect EU Council decisions)
- **voting weight** (number of votes)
- **voting power** held by a given state depends not only on its **voting weight** but also on the distribution of the **weights** among all the remaining states
- the **voting power** needs not to be proportional to the **voting weight**

Voting power vs. voting weight

the voting power needs not to be proportional to the voting weight !

- A simple example: shareholders' assembly takes decisions by a simple majority vote
  - shareholder ***X*** - 51% of stocks of a company (voting weight = 51%)
  - shareholder ***Y*** - 49% of stocks of a company (voting weight = 49%)

  - shareholder ***X*** - 100% of the voting power
  - shareholder ***Y*** - 0% of the voting power

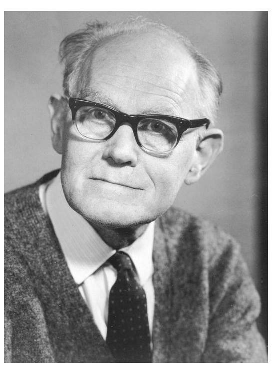
How to measure voting power?

- **power index** - probability that the vote of a country will be decisive in a hypothetical ballot measures the potential voting power

natural assumption:

all potential coalitions are equally likely

↓



Penrose-Banzhaf index

Indirect voting in the Council

- A representative of each country has to vote **yes** or **no** and cannot split his vote
- *example:* if **30** millions of Italians support a decision, and **29 M** are against, an Italian minister says **yes** (on behalf of **59** millions).
- Thus **30 M** Italians can overrule **39 M** Poles (+**29** millions of opposing Italians...)
- **One person-one vote** system would be perfect ... if all citizens of each country had the same opinion.

Is voting power important?

- **potential (*a priori*) voting power vs. actual voting power**
- ***value of stocks of a company -***

- How many stocks give an investor full control over the company?

(the answer depends on the distribution of the shares...)

- How much should he pay for them?

How to compute the Banzhaf index ?

(*Banzhaf*, 1965): **number of players** n

\# of coalitions 2^n

\# of winning coalitions w

\# of coalitions with ***i***-th player 2^{n-1}

\# of wining coalitions with ***i***-th player X_i w_i

\# of coalitions, for which the vote of X_i is critical

$$c_i := w_i - (w - w_i) = 2 \cdot w_i - w$$

Banzhaf index $= c_i / 2^{n-1}$

probability that vote of X_i will be decisive

Penrose-Banzhaf index (normalised)

$\beta_i = c_i / \sum_i c_i$ (*Penrose*, 1946): $p_i = (1 + \beta_i)/2$

probability, that player X_i is going to winn

Council of Ministers of **European Economic Community 1958-1972**

\# of countries: **$n = 6$**

sum of all votes (weights): S **$= 17$**

quota: **$q = 12$**

\# of coalitions $T = 2^6 = 64$

\# of coalitions with state X **32**

\# of winning coalitions: **$w = 14$**

State	votes	Winning coal. with X	Winning coal. Without	Diffe-rence	Banzhaf index	Banzhaf Normalis. Index
		w_i	$W - w_i$	c_i	$c_i/2^{n-1}$	β_i
Germany	4	12	2	10	5/16	5/21 ~ 0.24
France	4	12	2	10	5/16	5/21 ~ 0.24
Italy	4	12	2	10	5/16	5/21 ~ 0.24
Holland	2	10	4	6	3/16	3/21 ~ 0.14
Belgium	2	10	4	6	3/16	3/21 ~ 0.14
Luxemb.	1	7	7	0	0	0

Treaty of Nice

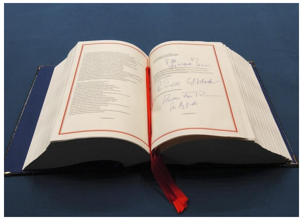
345 votes are distributed among ***27*** member states on a ***degressively proportional*** basis, e.g.:

- **DE, FR, IT, UK** – ***29*** votes (weight)
- **ES, PL** – ***27*** votes (weight), *etc.*

- a) the sum of the weights of the Member States voting in favour is at least ***255*** (~***73.9% of 345***)
- b) a majority of Member States (i.e. at least ***14*** out of ***27***) vote in favour
- c) the Member States forming the qualified majority represent at least ***62%*** of the overall population of the European Union

➢ **'triple majority'**

Constitutional Treaty



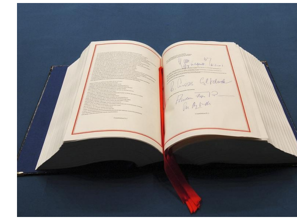
a) ------

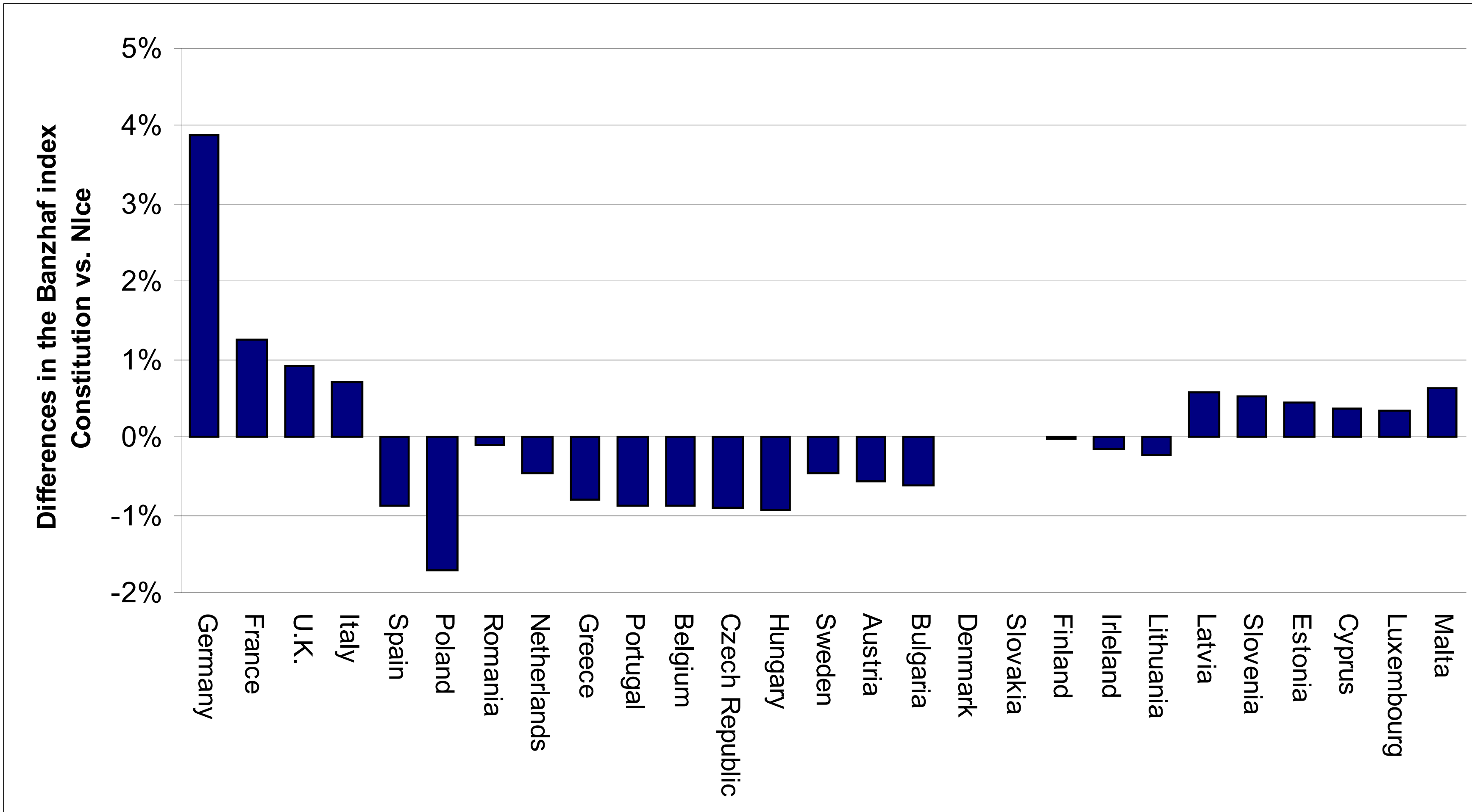
- b) at least ***55%*** of Member States (i.e. at least ***15*** out of ***27***) vote in favour
- c) the Member States forming the qualified majority represent at least ***65%*** of the overall population of the European Union

c') a blocking minority must include at least ***four*** Council members

➢ **'double majority'**

‘Nice’ vs. ‘Constitution’





(...) if two votings were required for every decision, one on a per capita basis and the other upon the basis of a single vote for each country, the system would be inaccurate in that it would tend to favour large countries.

[**L. Penrose, 1952**]

How the linear voting weights enhance the power of the largest states?

A model example: 160 M people living in one large state and 8 small. Assume that in both groups **51% of the population** votes **yes** in a certain case

- **group A):** One state with 80 M people
- **51%** say **yes**
- so does its minister in the Council
- result: **80M for**

- **group B):** 8 states with 10 millions each
- **51%** of people in this group say **yes, but majo-rity in each state varies**
- 8 ministers in the Council may vote as 4:4 (or 5:3...)
- result: **40M for** ***(less likely 50,60,70 or 80M)***

Penrose square root law:

Voting power of a citizen in a country with population ***N*** is proportional to ***N*** $^{-1/2}$

Bernoulli scheme for k=N/2 and p=q=1/2
+ Stirling expansion gives probability

$$P_k = p^k q^{1-k} \binom{N}{k}$$

$$P_k = \left(\frac{1}{2}\right)^{N/2} \left(\frac{1}{2}\right)^{N/2} \frac{N!}{(N/2)!(N/2)!} \approx$$

$$\approx \frac{1}{2^N} \frac{(N/e)^N \sqrt{2\pi N}}{\left[(N/2e)^{N/2} \sqrt{2\pi N/2}\right]^2} = \sqrt{\frac{2}{\pi N}} \sim \frac{1}{\sqrt{N}}$$

Square root weights - Penrose law

- this degressive system is distinguished by the ***Penrose square root law*** (1952)

$1/\sqrt{N}$ Voting power of a single citizen of a state with population **N**

$\sqrt{N}$ Voting power of its representative in EU Council

----

implies that

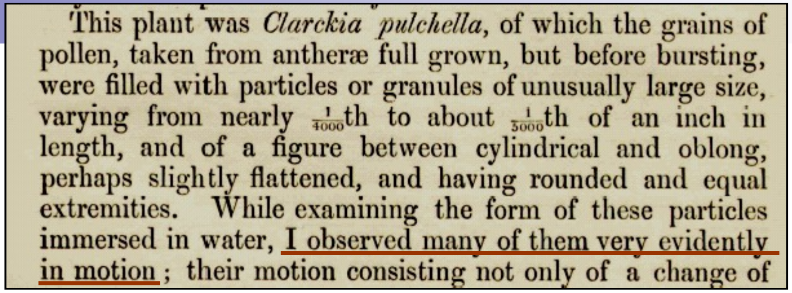
each citizen of each country has the same potential voting power !

Square root weights - example

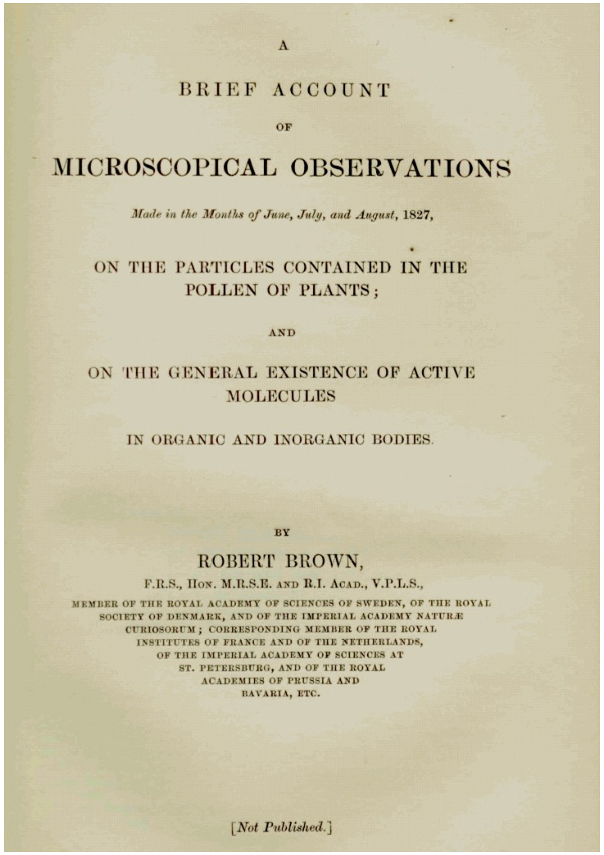
- **the ‘square root’ weights attributed to Member States are proportional to the sides of the squares representing their populations**

Brownian motion (1827)

- *Clarkia pulchella*



This plant was *Clarckia pulchella*, of which the grains of pollen, taken from antheræ full grown, but before bursting, were filled with particles or granules of unusually large size, varying from nearly $\frac{1}{4000}$th to about $\frac{1}{3000}$th of an inch in length, and of a figure between cylindrical and oblong, perhaps slightly flattened, and having rounded and equal extremities. While examining the form of these particles immersed in water, I observed many of them very evidently in motion; their motion consisting not only of a change of



A

BRIEF ACCOUNT

OF

MICROSCOPICAL OBSERVATIONS

Made in the Months of June, July, and August, 1827,

ON THE PARTICLES CONTAINED IN THE POLLEN OF PLANTS;

AND

ON THE GENERAL EXISTENCE OF ACTIVE MOLECULES

IN ORGANIC AND INORGANIC BODIES.

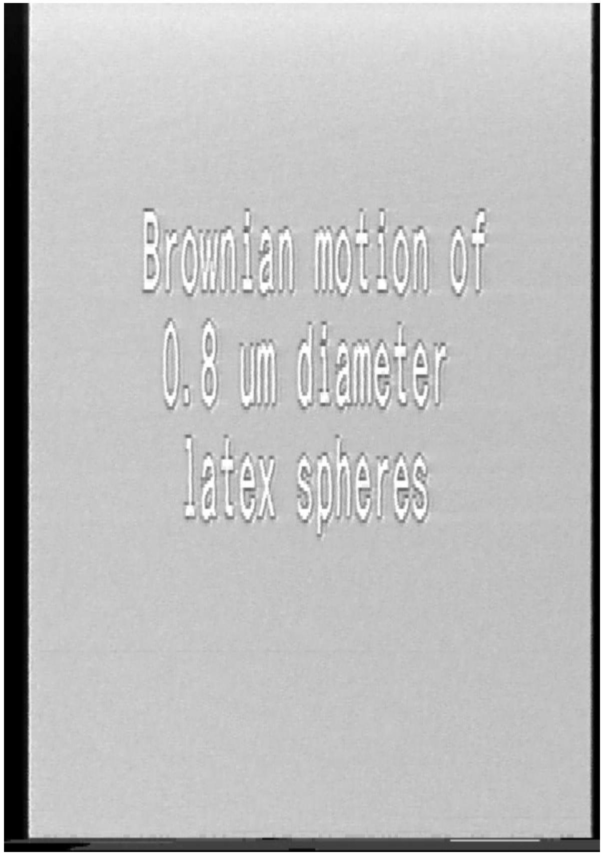
BY

ROBERT BROWN,

F.R.S., HON. M.R.S.E. AND R.I. ACAD., V.P.L.S.,

MEMBER OF THE ROYAL ACADEMY OF SCIENCES OF SWEDEN, OF THE ROYAL SOCIETY OF DENMARK, AND OF THE IMPERIAL ACADEMY NATURÆ CURIOSORUM; CORRESPONDING MEMBER OF THE ROYAL INSTITUTES OF FRANCE AND OF THE NETHERLANDS, OF THE IMPERIAL ACADEMY OF SCIENCES AT ST. PETERSBURG, AND OF THE ROYAL ACADEMIES OF PRUSSIA AND BAVARIA, ETC.

[*Not Published.*]



Albert Einstein (1879-1955)

Marian Smoluchowski (1872-1917)

5. ***Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen;***
von A. Einstein.

In dieser Arbeit soll gezeigt werden, daß nach der molekularkinetischen Theorie der Wärme in Flüssigkeiten suspendierte Körper von mikroskopisch sichtbarer Größe infolge der Molekularbewegung der Wärme Bewegungen von solcher Größe ausführen müssen, daß diese Bewegungen leicht mit dem Mikroskop nachgewiesen werden können. Es ist möglich, daß die hier zu behandelnden Bewegungen mit der sogenannten „Brownschen Molekularbewegung" identisch sind; die mir erreichbaren Angaben über letztere sind jedoch so ungenau, daß ich mir hierüber kein Urteil bilden konnte.

XXIX. ZARYS KINETYCZNEJ TEORJI RUCHÓW BROWNA I ROZTWORÓW MĘTNYCH.

(Rozprawy Wydziału matematyczno-przyrodniczego Akademji Umiejętności w Krakowie. T. XLVI. Serja A. 1906; str. 257—281).

§ 1. Ruch, polegający na dygotaniu i trzęsieniu się, który odbywają drobne, w silnem powiększeniu jeszcze widzialne cząstki, znajdujące się w stanie zawieszenia w cieczach, były często badane od r. 1827, w którym zwrócił na nie uwagę botanik Robert Brown, aż do dziśdnia; a jednak zjawisko to nie zostało jeszcze dostatecznie objaśnione. Żadna z pomiędzy różnych proponowanych teoryj nie przyjęła się powszechnie. Niepewność ta pochodzi częściowo z nie-

Marian Smoluchowski (1906)

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Smoluchowski's explanation of the Brownian motion...

Stąd znajdujemy wartość przeciętnego zboczenia w jednę lub drugą stronę:

$$\nu = 2\sum_{m=\frac{n}{2}}^{n} \frac{2m-n}{2^n}\binom{n}{m},$$

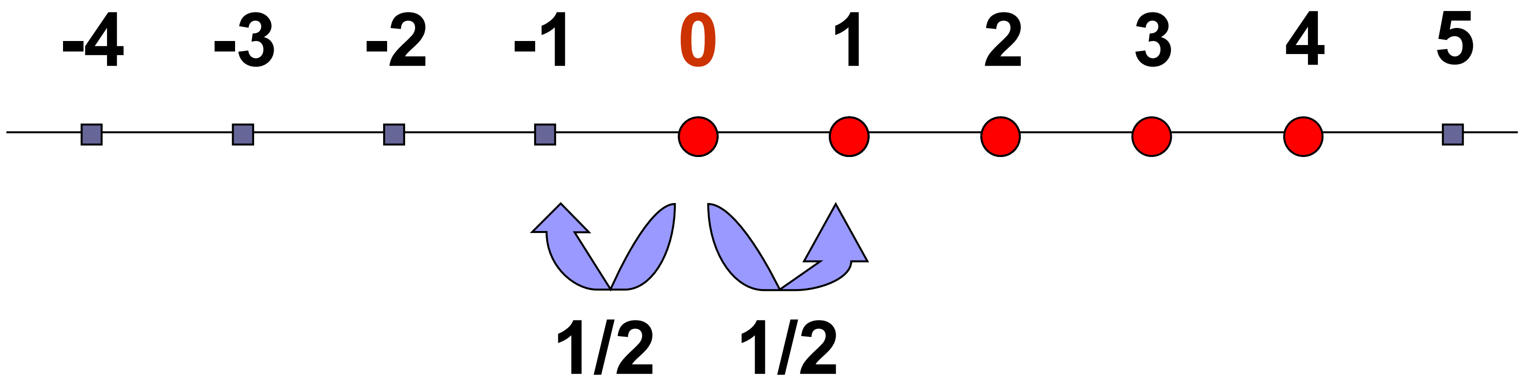
jeżeli dla uproszczenia liczbę n przyjmiemy za parzystą. Wyrażenie to można przekształcić przez zastosowanie twierdzenia dwumianowego w formę dogodniejszą:

(1) $$\nu = \frac{n}{2^n}\binom{n}{\frac{n}{2}},$$

która dla dużych liczb n przechodzi w

(2) $$\nu = \sqrt{\frac{2n}{\pi}}.$$

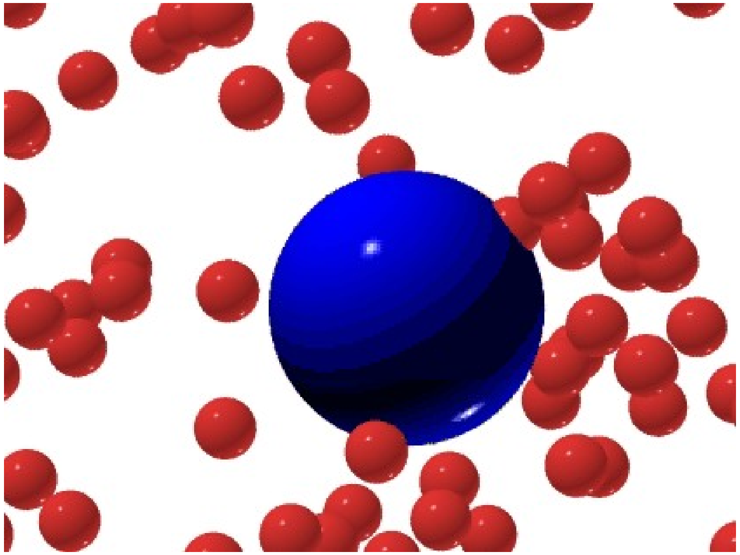
Random walk on the line



- If particle moves from point different then **0** its **mean** distance to **0** does not change
- if particle moves from the point **0** its **mean** distance to **0** grows by 1

Random walk : a diffusion law

- **Probability that a particle returns to its initial point after k steps scales as $k^{-1/2}$**
- **Thus the mean distance <Dx> from 0 grows with the time n as**



$$\langle Dx(n) \rangle \sim \Sigma_{k=1}^{n} \; k^{-1/2} \;\sim n^{1/2}$$

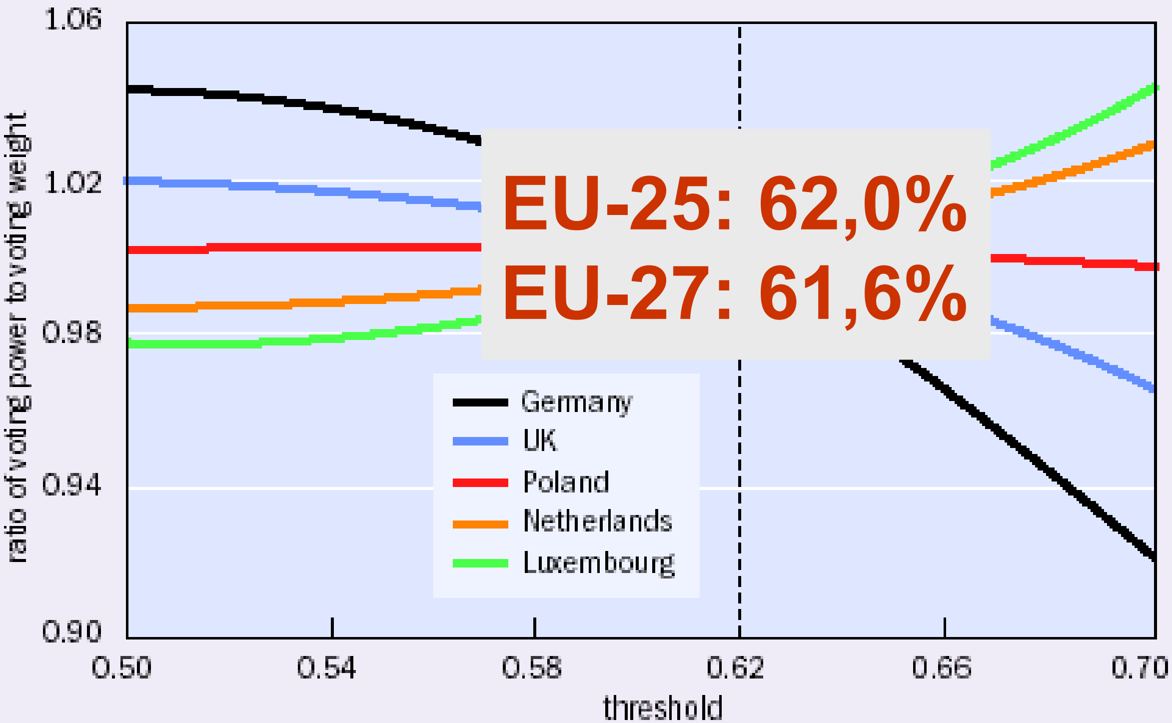
The Penrose square root law is closely related to diffusion law !

Qualified majority threshold

- The choice of an appropriate decision-taking **quota** (**threshold**) affects both the distribution of voting power in the Council (and thus also the representativeness of the system) and the voting system’s effectiveness and transparency.
- Different authors have proposed different quotas for a square root voting systems, usually varying from 60% to 74%.
- The **optimal quota** enables the computed voting power of each country to be practically equal to the attributed voting weight.

Optimal threshold

1 The critical point



EU-25: 62,0%

EU-27: 61,6%

By plotting the ratio of the voting power of each country to its voting weight against the threshold chosen for majority, a "critical point" emerges. At a threshold of 62% each country achieves a ratio of one, meaning its voting power is equal to its voting weight. Since the weights are chosen to be proportional to the square root of population, the voting power of each citizen is equal. With a lower threshold, larger countries have disproportionate power, while for a higher threshold, smaller countries have more influence.

Jagiellonian Compromise

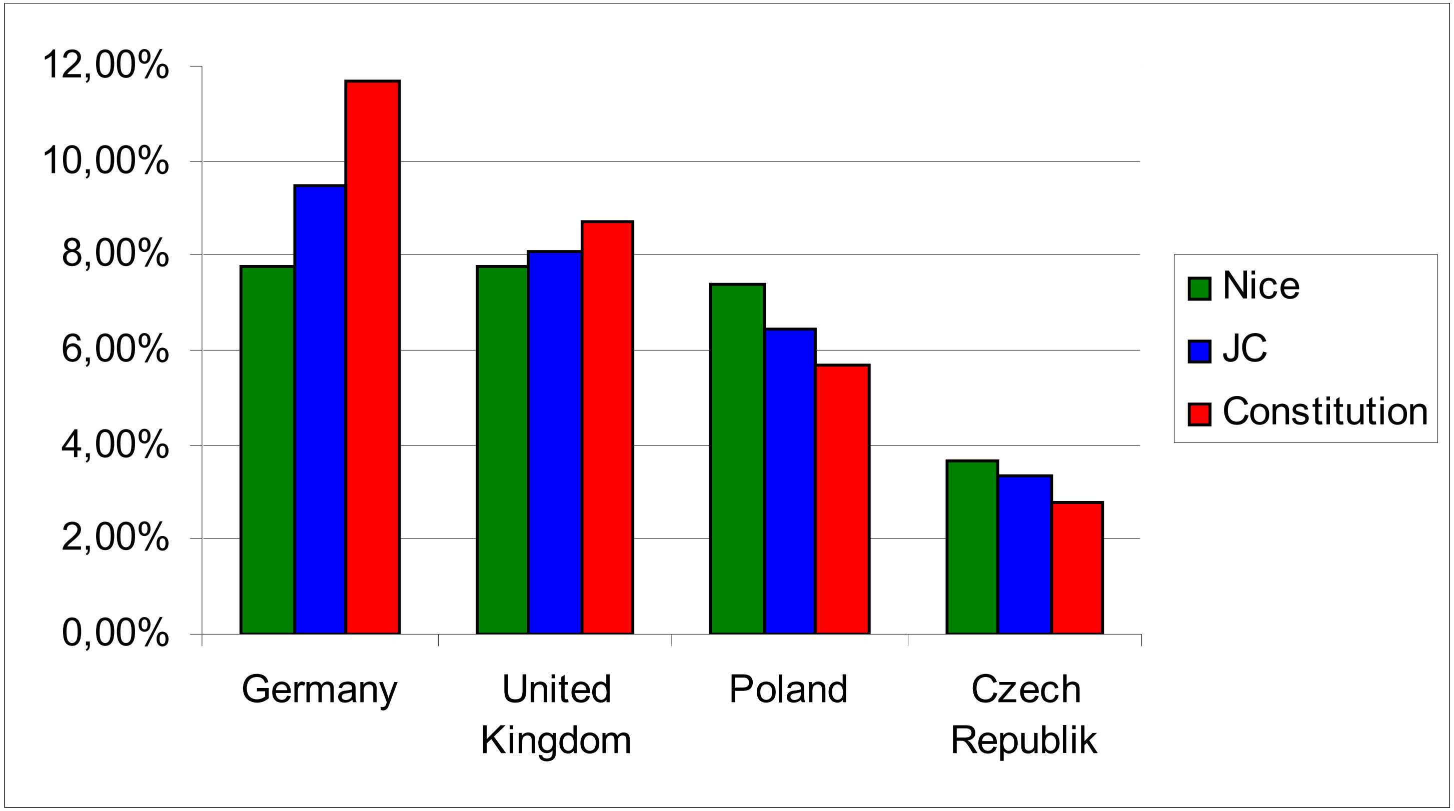
↓

square root weights

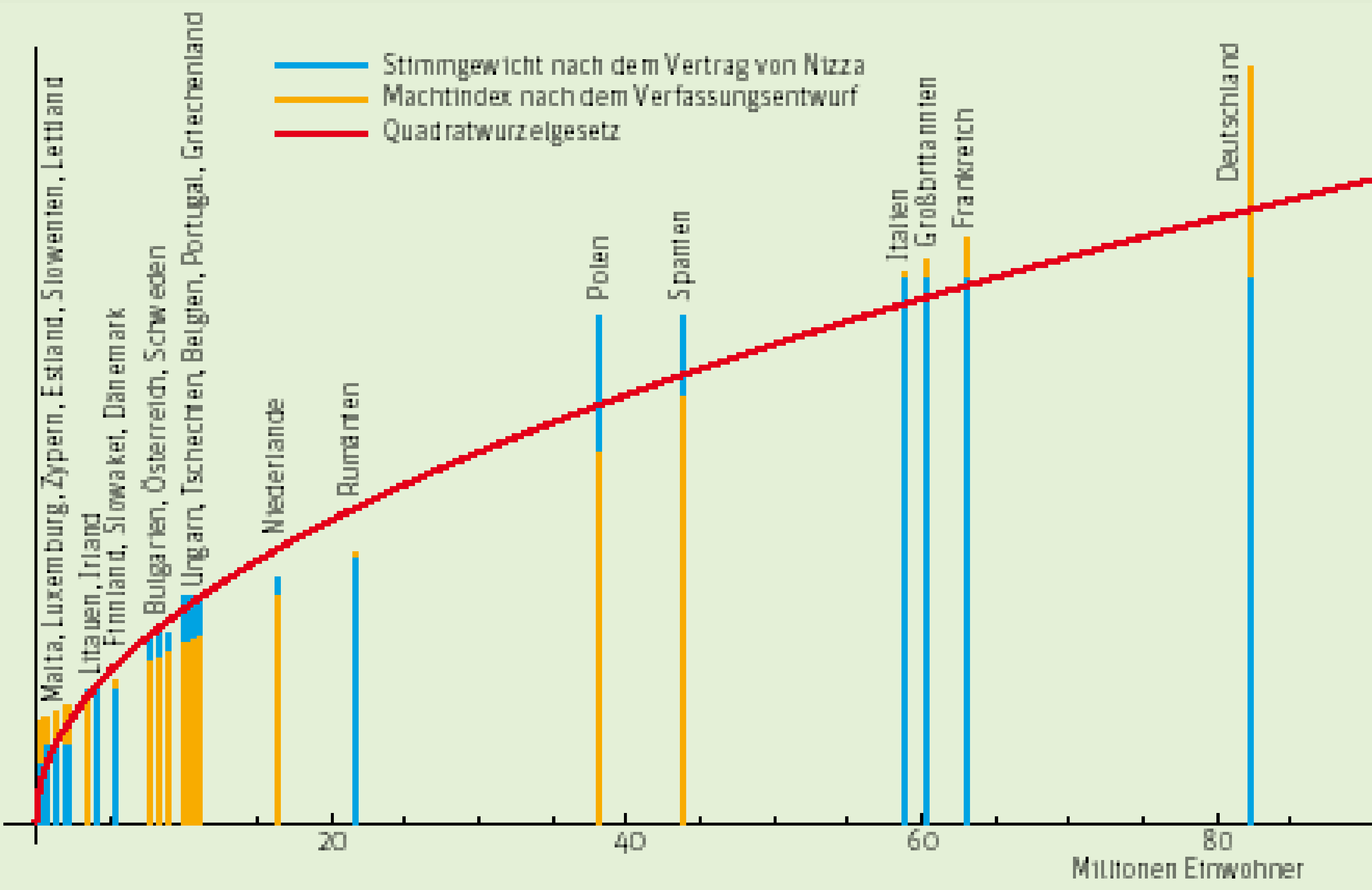
+

optimal quota

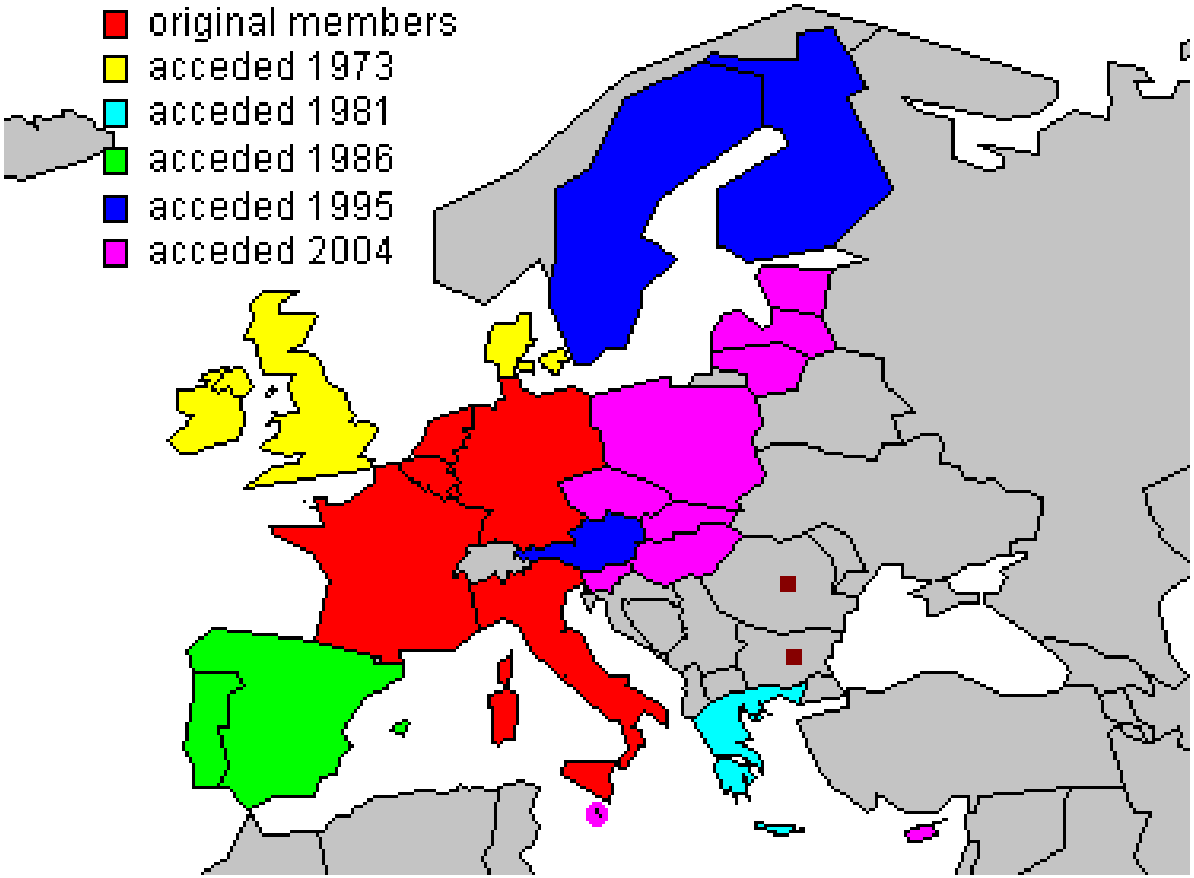
'Nice' / Jagiellonian Compromise / Constitution



Member State	Population (in millions)	Voting power Constitution	Voting weight (JC)	Voting power (JC)
Germany	82.44	11.66	9.47	9.45
France	62.89	9.02	8.27	8.27
United Kingdom	60.39	8.69	8.10	8.10
Italy	58.75	8.49	7.99	7.99
Spain	43.76	6.55	6.90	6.91
Poland	38.16	5.71	6.44	6.45
Romania	21.61	4.15	4.85	4.85
Netherlands	16.33	3.50	4.21	4.21
Greece	11.13	2.88	3.48	3.48
Portugal	10.57	2.80	3.39	3.39
Belgium	10.51	2.80	3.38	3.38
Czech Rep.	10.25	2.77	3.34	3.34



From Christoph Poeppe, Spektrum der Wissenschaft 2007



EU-M	6	9	10	12	15	25	27
R_{opt}	73.0%	67.4%	65.5%	64.5%	64.4%	62,0%	61.4%

Tab. 2 shows the value of the critical quota R_{opt} as a function of the number M of members of the EU.

Optimal quota – the normal approximation

$w_k \ (k=1,\dots,M)$ - *voting weights,* $\sum_{i=1}^{k} w_i = 1,\ [q; w_1,\dots,w_M]$

$$n(z) := \frac{card\left\{ I \subset \{1,\dots,M\} : \sum_{i\in I} w_i = z \right\}}{2^M}, \quad N(q) := \sum_{z\le q} n(z) \approx \Phi\left(\frac{q-m}{\sigma}\right)$$

$m := \frac{1}{2}\sum_{i=1}^{k} w_i = \frac{1}{2},\ \sigma^2 := \frac{1}{4}\sum_{i=1}^{k} w_i^2,\quad \Phi$ - *standard normal cumulative distribution function*

$$\psi_k(q) \approx \Phi\left(\frac{q-m+w_k/2}{\sqrt{\sigma^2 - w_k^2/4}}\right) - \Phi\left(\frac{q-m-w_k/2}{\sqrt{\sigma^2 - w_k^2/4}}\right) \qquad (k=1,\dots M)$$

$$q_n := m+\sigma = \frac{1}{2}\left(1+\sqrt{\sum_{i=1}^{k} w_i^2}\right)$$

$$\psi_k(q_n) \approx \sqrt{\frac{2}{\pi e}}\, v_k + o\left(v_k^4\right), \quad v_k := \frac{w_k}{\sqrt{\sum_{i=1}^{k} w_i^2}} << 0 \quad \leftarrow ASSUMPTION$$

$$\beta_k(q_n) \approx w_k$$

$q = m = 1/2$

$$\psi_k(1/2) \approx \sqrt{\frac{2}{\pi}}\, v_k + o\left(v_k^2\right)$$

Optimal quota –solution of the problem

$w_k \ (k = 1,\dots,M)$ *- voting weights,* $\sum_{i=1}^{k} w_i = 1,\ [q; w_1,\dots,w_M]$

$$q_* \cong q_n(w_1,\dots,w_M) = \frac{1}{2}\left(1 + \sqrt{\sum_{i=1}^{k} w_i^2}\right)$$

$$q_s(M) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{M}}\right) \le \frac{1}{2}\left(1 + \frac{1}{\sqrt{M_{eff}}}\right) = q_n(w_1,\dots,w_M),$$

where $M_{eff} := \dfrac{1}{\sqrt{\sum_{i=1}^{k} w_i^2}}$ *- effective number of players (Laakso, Taagepera (1979))*

In particular for Penrose voting system $(w_k \sim \sqrt{N_k})$ *we get*

$$q_n(N_1,\dots,N_M) = \frac{1}{2}\left(1 + \frac{\sqrt{\sum_{i=1}^{k} N_i}}{\sum_{i=1}^{k} \sqrt{N_i}}\right)$$

The efficiency of the system does not decrease, when the number of players M increases

$$A(q_s) \ge A(q_n) \approx 1 - \Phi(1) \approx 15.9\%$$

- For the **Council of Ministers of EU-27** the optimal quota equals **61.6%**.
- For **EU-M** the optimal quota ***q*** can be approximated by a simple mathematical formula:

$$q = \frac{1}{2}\left(1 + \frac{\sqrt{N_1 + \ldots + N_M}}{\sqrt{N_1} + \ldots + \sqrt{N_M}}\right)$$

where N_i stands for the population of the *i*-th country.

Jagiellonian Compromise

- it is extremely **simple** since it is based on a single criterion, and thus it could be called a '**single majority**' system;
- it is **objective** (no arbitrary weights or thresholds), hence cannot *a priori* handicap any member of the European Union;
- it is **representative**: every citizen of each Member State has the same potential voting power;
- it is **transparent**: the voting power of each Member State is (approximately) proportional to its voting weight;
- it is **easily extendible**: if the number of Member States changes, all that needs to be done is to set the voting weights according to the square root law and adjust the quota accordingly;
- it is **moderately efficient**: as the number of Member States grows, the efficiency of the system does not decrease;
- it is also **moderately conservative**, that is, it does not lead to a dramatic transfer of voting power relative to the existing arrangements.

Square root weights
- support from academics

- advocated or analysed by Laruelle, Widgrén (1998), Baldwin, Berglöf, Giavazzi, Widgrén (2000), Felsenthal, Machover (2000-2004), Hosli (2000), Sutter (2000), Tiilikainen, Widgrén (2000), Kandogan (2001), Leech (2002), Moberg (2002), Hosli, Machover (2002), Leech, Machover (2003), Widgrén (2003), Baldwin, Widgrén (2004), Bilbao (2004), Bobay (2004), Kirsch (2004), Lindner (2004), Lindner, Machover (2004), Plechanovová (2004, 2006), Sozański (2004), Ade (2005), Koornwinder (2005), Pajala (2005), Maaser, Napel (2006), Taagepera, Hosli (2006)
- prior to the European Union summit in Brussels in June 2004, an open letter in support of square-root voting weights in the Council of Ministers endorsed by more than 40 scientists from 10 European countries

Der Jagiellonische Kompromiss

Polen und ein neues Abstimmungssystem für den EU-Ministerrat

Von Friedrich Pukelsheim*

Polen wirkt in der Europäischen Union darauf hin, das Abstimmungssystem im Ministerrat gegenüber dem in Volksabstimmungen abgelehnten Verfassungsvertrag und dem Vertrag von Nizza zu verbessern. Der Autor liefert einen sachlichen Beitrag auch für ein eminent politisches Problem, das endemische Demokratiedefizit der EU.

Die Namensgebung ist kein Kotau der polnischen Presse vor ihrer Regierung, sondern honoriert den Beitrag von Wissenschaftern der Jagiellonischen Universität in Krakau. Das von der polnischen Regierung vorgeschlagene System ist schon ein halbes Jahrhundert alt, wurde aber von der polnischen Presse erst jüngst mit dem geschichtsträchtigen Namen Jagiellonischer Kompromiss versehen. Der Jagiellonische Kompromiss garantiert allen Bürgern und Bürgerinnen der Union, dass ihnen über ihre Mandatsträger der gleiche Einfluss auf Ministerratsbeschlüsse zukommt. Er hilft, das Demokratiedefizit in der Europäischen Union zu vermindern, und stärkt die Legitimationskraft der Entscheidungen des Ministerrats.

Transparenz und Einfachheit

Zudem ist der Jagiellonische Kompromiss das transparenteste unter allen gewichteten Abstimmungssystemen, die bisher verwendet oder vorgeschlagen wurden. Die Stimmenzahlen der Mitgliedstaaten werden nicht in nächtelangen Verhandlungsmarathons ausgepokert, sondern be-

Stimmengewichte der Mitgliedstaaten (EU-27)

	Bevölkerung	Stimmen	Machtanteil in %
Deutschland	82 311 600	9 073	9,44
Frankreich	63 336 200	7 958	8,28
Grossbritannien	60 707 100	7 791	8,11
Italien	58 938 800	7 677	7,99
Spanien	44 484 300	6 670	6,94
Polen	38 101 800	6 173	6,42
Rumänien	21 590 600	4 644	4,83
Niederlande	16 346 200	4 043	4,21
Griechenland	11 169 100	3 342	3,48
Portugal	10 609 000	3 257	3,39
Belgien	10 570 500	3 251	3,38
Tschechische Republik	10 288 900	3 208	3,34
Ungarn	10 057 900	3 171	3,30
Schweden	9 119 800	3 020	3,14
Österreich	8 295 900	2 880	3,00
Bulgarien	7 666 500	2 769	2,88
Dänemark	5 445 700	2 334	2,43
Slowakei	5 391 600	2 322	2,42
Finnland	5 277 100	2 297	2,39
Irland	4 328 700	2 080	2,16
Litauen	3 385 700	1 840	1,91
Lettland	2 280 600	1 510	1,57
Slowenien	2 010 300	1 418	1,48
Estland	1 339 900	1 158	1,21
Zypern	776 000	881	0,92
Luxemburg	464 400	681	0,71
Malta	407 300	638	0,67
Total	**494 674 800**	**96 087**	**100,00**
Quorum		59 185	61,57

Der Jagiellonische Kompromiss. Die Stimmengewichte sind die gerundeten Wurzeln der Bevölkerungszahlen (am 1. 1. 2007, aus Eurostat 41/2007). Für die Gültigkeit eines Beschlusses braucht es davon mindestens 59 185 [= (√494 674 800 + 96 087) / 2]. Bei diesem Quorum

sollte. Man mag die obige Argumentation kritisieren, aber ihre Grundlagen und Schlussfolgerungen haben sich bisher theoretisch wie praktisch hervorragend bewährt. Als erster Wissenschafter studierte der englische Psychiater Lionel Penrose (1898–1972) Abstimmungssysteme für die Uno-Generalversammlung. Das Schrifttum spricht deshalb vom Quadratwurzel-Gesetz von Penrose und – statt Entscheidungsmacht – vom normalisierten Penrose-Index. Dieser Index wird auch nach dem Juristen John Banzhaf benannt, der zwanzig Jahre später das Thema neu aufgriff. Seither sind Abstimmungssysteme mit all ihren Verästelungen eingehend untersucht worden.

Die Altmeister des Faches, der Philosoph Moshe Machover (London) und der Politikwissenschafter Dan Felsenthal (Haifa), haben zudem ergänzende Facetten beleuchtet, zum Beispiel die Sensitivität, mit welchen Erfolgsaussichten ein Mitgliedstaat eine Beschlussinitiative starten kann, oder die Resistenz, mit welchen Aussichten er die Initiative anderer abblocken kann. Alle ihre Studien führen zu dem System, das jetzt unter dem Namen Jagiellonischer Kompromiss firmiert.

Das Tüpfelchen auf dem i haben vor drei Jahren der Mathematiker Wojciech Slomczynski und der Physiker Karol Zyczkowski aus Krakau beigesteuert. Sie zeigten, dass die Wurzel der Gesamtbevölkerung und die Gesamtsumme der Bevölkerungswurzeln sich zum optimalen Quorum ausmitteln. Das spröde akademische Optimum wurde von polnischen Journalisten konziliant umschrieben als Jagiellonischer Kompromiss.

Nicht nur gut für Polen


Reform Treaty

Treaties are like roses and young girls. They last while they last.

Charles de Gaulle

Time, 12th July, 1963

Spektrum der Wissenschaft
August 2007

Die Quadratwurzel, das Irrationale und der Tod

Christoph Pöppe ist Redakteur bei Spektrum der Wissenschaft.

Optimal quota for the union of M states:

$$q_{opt}=(1/2+M^{-1/2}/2).$$