RANDOM VARIABLE - BASIC CONCEPTS

1. A drunk wanders out of a bar thoroughly inebriated. At each step he will move to the right one (1) meter with probability p or to the left one meter with $\mathcal{P} = 1-p$. How many meters from the entrance to the bar do you expect the drunk to be after n steps?

   X_n – # of meters after n steps; $E(X)$ of a single step is (two-way RV)
   $E(X) = 1 \times p + (-1) \times (1-p) = 2p - 1$.

   $E(X_n) = \sum_{i=1}^{n} E(X) = \sum_{i=1}^{n} 2p - 1 = n(2p-1)$.

   Mark: for $p = 1/2 \quad E(X_n) = 0$

2. Show that $E(aX) = aE(X)$ if a is a constant and X is a continuous or discrete RV (consider both cases).

   Show also $VAR(aX) = a^2 VAR(X)$.

   $VAR(aX+b) = E\left([(aX+b)^2]\right) - [E(aX+b)]^2 = a^2E(X)^2 + 2abE(X) + b^2 - a^2E(X)^2 - 2abE(X) - b^2 = a^2E(X)^2 - a^2E(X)^2 = a^2VAR(X)$.

   Once you perform the necessary (and simple) calculations think – were they really necessary? The results can be justified by simple argumentation.

3. A sample consists of results: $2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 7$. Compute: (a) μ; (b) σ^2.

   4.5 AND 1.9

4. A discrete random variable X has the probability function :

x_i	-2	-1	2	5
$P(X = x_i)$	0.3	0.1	0.2	0.4

   (a) sketch the cumulative distribution of X ; (b) calculate the mean and variance for two random variables: $U = 2X - 3$ and $V = 2X^2$. Comment on $p(U = u_i)$ and $p(V = v_k)$ values.

   $E(U) = 0.4; \quad VAR(U) = 36.84; \qquad E(V) = 24.2; \quad VAR(V) = 446.76$

   mark: in the case of V-variable the two *different* values of X $(-2,\ 2)$ are mapped into a single V (8). Thus:

$$\mathcal{P}(V = 8) = \mathcal{P}(X = -2) + \mathcal{P}(X = -2)$$

5. The probability density function of the random variable X is given by:

$$f(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 0 \\ \sin x & \text{for} \quad 0 \le x \le \frac{1}{2}\pi \\ 0 & \text{for} \quad \frac{1}{2}\pi < x < \infty \end{cases}$$

   Find the cumulative distribution $F(x)$ and calculate $P(1/6\pi < x < 1/3\pi)$.

Sketch the graphs of probability density function and cumulative distribution function and read out the calculated P from the graphs.

$$F(x) = \int_0^x f(x)\,dx = 1 - \cos x \quad x \leq \frac{1}{2}\pi.$$

$\mathcal{P}(1/6\pi < x < 1/3\pi) = \frac{1}{2}(\sqrt{3} - 1)$

6. A continuous random variable has a *uniform* distribution, i.e. its probability density function is constant in the whole interval. For example

$$f(x) = C \qquad \text{for} \quad x \in [0., 2.]$$

(a) find the value of the constant C.

(b) Calculate the expected value $E(X)$ and variance $VAR(X)$.

$C = 1/2; \quad E(X) = 1; \quad VAR(X) = 1/3$

7. Given that the RV X has the probability density function

$$f(x) = \frac{1}{2}(x+1) \quad \text{when} \quad -1 < x < 1$$

and $f(x) = 0$ elsewhere, calculate the expected value of X and the variance of X.

1/3 and 2/9

8. Given that the RV X has the density probability function

$$f(x) = \frac{1}{2\sqrt{3}} \quad \text{when} \quad -\sqrt{3} < X < \sqrt{3}$$

and $f(x) = 0$ elsewhere, compute $\mathcal{P}\left(|X| \geq \frac{3}{2}\right)$ and $\mathcal{P}\left(|X| \geq 2\right)$. Compare these probabilities with the upper bounds given by the Chebyshev's inequality.

We have $\mu = 0$ (by symmetry); and $\sigma = \ldots = 1$; $\qquad \sigma^2 = L^2/12 = 4 \times 3/12 = 1$

We have $\mathcal{P}\left(|X| \geq \frac{3}{2}\right) \leq \frac{1}{(3/2)^2} = \frac{4}{9}$ (Chebyshev) and $1 - \sqrt{3}/2 \approx 0.134$ (computations).

The second question: $\mathcal{P}\left(|X| \geq 2\right) \leq \frac{1}{(1/2)^2} = \frac{1}{4}$ (Chebyshev) but the real $\mathcal{P}$ is zero! (2 is outside the interval!)

9. Use the Chebyshev's inequality to find the lower bound on $\mathcal{P}(-4 < X < 20)$, where X is a RV with mean $\mu = 8$ and variance $\sigma^2 = 9$.

In accordance with the Chebyshev's inequality: $\mathcal{P}(X \geq 20 \quad \text{or} \quad X \leq -4) \leq 1/4^2 = 1/16$; hence $\mathcal{P}(-4 < X < 20) \geq 15/16$.

10. If X follows a *discrete uniform distribution* i.e. $\mathcal{P}(X = i) \quad i = 1, 2, \ldots, N-1, N = \frac{1}{N}$ find $E(X)$ and $\sigma^2(X)$.

$\mu = (N+1)/2$ and $\sigma^2 = (N^2 - 1)/12$

Hint: use the formula $\sum_{X=1}^{N} X^2 = \frac{N(2N+1)(N+1)}{6}$ (by the way, would you know how to justify this formula?).