Some important distributions – discrete case

1. An absent-minded professor has a bunch of 5 keys. One of these opens the door to his apartment. His usual strategy is: he selects a key at random – and tries it in the lock. If it does not work he replaces the key and selects another one from the whole bunch. Let X denote the number of attempts the prof makes. What is the probability function of X – $f(x)$?

   It's quite easy to see:

$$f(k) = \left(\frac{4}{5}\right)^{k-1} \frac{1}{5}, \quad \text{where } k \text{ is the number of the last attempt.}$$

   The sum (geometric distribution)

$$\sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^{k-1} \frac{1}{5} = \frac{1}{5} \sum_{k=0}^{\infty} \left(\frac{4}{5}\right)^{k} = \ldots = 1.$$

2. X is a discrete random variable (with an infinite but countable # of values) with probability function

$$\mathcal{P}(X = x) = f(x) = \left(\frac{1}{2}\right)^{x}; \qquad x = 1, 2, \ldots.$$

   Find the probability that X is even.

$$\mathcal{P}(X \text{ is even}) = \mathcal{P}(X = 2) + \mathcal{P}(X = 4) + \ldots + \mathcal{P}(X = 2n) + = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n} = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}.$$

   The last sum may be converted

$$\frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} = \ldots = 1/3.$$

3. The deck of cards can dichotomized into hearts (13) and all other cards (39). What is the p of getting a heart on a single draw (1/4). What is probability q of getting getting a spade OR club OR diamond (3/4). When 7 cards are sampled **with replacement** what is the $\mathcal{P}$ of: a) getting no hearts at all?; b) getting 4 hearts? What is the $\mathcal{P}$ of getting 2 hearts out of the first four draws and then 2 hearts out of the next 3? Is the last result more or less probable than '4 hearts out of 7'? Why?

   the second $\mathcal{P}$ is less because the scheme restrics the # of ways the total of 4 can be arranged.

4. The most common application of the binomial theorem in industrial work is in in lot-by-lot acceptance inspection. If there are a certain # of defectives the lot will be rejected.

   It's natural to wish to find the $\mathcal{P}$ that the lot is acceptable even though a certain # of defectives are observed. Let p be the fraction of defectives. We assume that the size of the sample is **small** compared to the lot size. This keeps the p (almost) constant.

Let's now choose a sample from a lot where 10% of the items are defective. What is the $\mathcal{P}$ of observing 0, 1 or 2 defectives in the sample. Assume the sample size 18.

$$\mathcal{P} = \binom{18}{0}(0.1)^0(0.8)^{18} + \binom{18}{1}(0.1)^1(0.8)^{17} + \binom{18}{2}(0.1)^2(0.8)^{16} = \ldots = 0.734$$

Note: such a sampling follows rather the so-called hypergeometric distribution (cf. lecture 'Some most popular distributions of RVs'))

5. The $\mathcal{P}$ of hitting a target on a shot is 2/3. A person fires 8 shots; the total # of hits is X. Find:

$$\mathcal{P}(X = 3); \quad \mathcal{P}(1 < X \leq 6); \quad \mathcal{P}(X > 3).$$

Solution: cf. the next problem

6. The probability that a certain kind of component will survive a given shock test is 0.75.
Find the probability that:
(a) exactly 2 of the next 4 components tested survive
(b) at least 2 of the next 4 components tested survive.(a)

$$W_k^n(n,p) \equiv W_2^4(4, 0.75) = \binom{4}{2} 0.75^2\, 0.25^2 = 27/128$$

(b)

$$\sum_{k=2}^{4} W_k^n(n,p) = 1 - \sum_{k=0}^{1} W_k^n(n,p) = 1 - W_0^4(4, 0.75) - W_1^4(4, 0.75)$$

7. Probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease what is the probability that: (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive? (a)

$$P(X \geq 10) = 1 - P(X < 10) = 1 - \sum_{k=0}^{9} W_k^{15}(15, 0.4) = \text{tables} = 1 - 0.9662;$$

(b)

$$\begin{aligned} P(3 \leq X \leq 8) &= \sum_{k=3}^{8} W_k^{15}(15, 0.4) = \sum_{k=0}^{8} W_k^{15}(15, 0.4) - \sum_{k=0}^{2} W_k^{15}(15, 0.4) \\ &= \text{tables} = 0.9050 - 0.0271 = 0.8779 \end{aligned}$$

(c) 0.12859

8. Calculate the expected value μ and the standard mean deviation σ for the number of patients to recover in the above example. Use the Chebyshev's theorem to find the range in which the number of the recovered patients should fall with 75% probability.
Answer: *between 3 and 9 – expressing results as integers* $E(X) = np = 6; \quad \sigma^2 = npq = 3.6$
2-sigma interval $\pm 2, 2.$

9. The average number of radioactive particles passing a counter during 1 millisecond in a laboratory experiment is 4. What is the probability that 6 particles enter the counter in a given millisecond?
$P_6(\lambda = 4) = e^{-4}\frac{4^6}{6!} = 0.1042$

10. The average number of oil tankers arriving each day at a port is known to be 10. The facilities of the port can handle at most 15 tankers per day. What is the probability that on a given day tankers will have to be sent away?

    $P(X > 15; \lambda = 10) = \sum_{k=16}^{\infty} P_k = 1 - \sum_{k=0}^{15} P_k =$ tables $= 1 - 0.9513 = 0.0487$

11. $\lambda = np = 8000 \times 0.001 = 8$

    fewer than seven means 0, or 1, or $\ldots$, or 6

12. On the average a certain intersection results in 3 traffic accidents per month. What is the probability that in a given month at this intersection:
    (a) exactly 5 accidents will occur?
    (b) less than 3 accidents will occur?
    (3) at least 2 accidents will occur? (b) $P(X < 5) = \sum_{k=0}^{4} P_k(\lambda = 4) =$ tables $= \ldots$

13. The probability that a person dies from a certain respiratory infection is 0.002. Find the probability that fewer than 5 of the next 2000 so infected will die. Find the mean and variance of the RV representing the number of persons among 2000 that die from the respiratory infection. According to Chebyshev's theorem, there is a probability of at least 3/4 that the number of persons to die among 2000 persons infected will fall within what interval?

    $\sigma = \sqrt{4} = 2$ Chebyszev's 4-sigma interval: 0-8